Opérateur hamiltonien

Exercices corrigés

Dans tous les exercices suivants, on désigne par

$$\overrightarrow{r} = \overrightarrow{OM} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$$

le rayon vecteur d'un point $M(x,y,z)$ de l'espace rapporté au système d'axes Ox, Oy, Oz et

$$r = |\overrightarrow{r}| = \sqrt{x^2+y^2+z^2} \neq 0$$

Exercice 1

Trouver le gradient et le laplacien du champ scalaire $f = f(M)$ dans les cas suivants :

1. $f(x,y,z) = xy + xz + yz$
2. $f(x,y,z) = xy^2z^3$
3. $f(x,y,z) = \ln(x^2+y^2+z^2)$
4. $f(x,y,z) = e^{-xyz}$
5. $f(x,y,z) = e^{ax+by+cz}$
6. $f(x,y,z) = (x^2+y^2+z^2)^n, n \in \mathbb{Z}$.

Solution 1 : On a $\overrightarrow{\nabla} f = \dfrac{\partial f}{\partial x}\overrightarrow{i} + \dfrac{\partial f}{\partial y}\overrightarrow{j} + \dfrac{\partial f}{\partial z}\overrightarrow{k}$ et $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

1. $f(x,y,z) = xy + xz + yz$

$$\frac{\partial f}{\partial x} = y + z \Longrightarrow \frac{\partial^2 f}{\partial x^2} = 0$$

$$\frac{\partial f}{\partial y} = x + z \Longrightarrow \frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial f}{\partial z} = x + y \Longrightarrow \frac{\partial^2 f}{\partial z^2} = 0$$

$$\overrightarrow{\nabla} f = (y+z)\,\overrightarrow{i} + (x+z)\,\overrightarrow{j} + (x+y)\,\overrightarrow{k} \quad \text{et } \Delta f = 0$$

2. $f(x,y,z) = xy^2z^3$

$$\frac{\partial f}{\partial x} = y^2z^3 \qquad \frac{\partial f}{\partial y} = 2xyz^3 \qquad \frac{\partial f}{\partial z} = 3xy^2z^2$$

$$\overrightarrow{\nabla} f = y^2z^3\,\overrightarrow{i} + 2xyz^3\,\overrightarrow{j} + 3xy^2z^2\,\overrightarrow{k}$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 2xz^3 \qquad \frac{\partial^2 f}{\partial z^2} = 6xy^2z$$

$$\Delta f = 2xz^3 + 6xy^2z = 2xz\left(z^2+3y^2\right)$$

3. $f(x,y,z) = \ln(x^2+y^2+z^2)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\ln\left(x^2+y^2+z^2\right)\right) = \frac{2x}{x^2+y^2+z^2}$$

de même : $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2+z^2}$ et $\dfrac{\partial f}{\partial z} = \dfrac{2z}{x^2+y^2+z^2}$

$$\overrightarrow{\nabla} f = \frac{2x\,\overrightarrow{i} + 2y\,\overrightarrow{j} + 2z\,\overrightarrow{k}}{x^2+y^2+z^2} = \frac{2\,\overrightarrow{r}}{r^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\frac{2x}{x^2+y^2+z^2} = 2\frac{-x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}$$

de même : $\frac{\partial^2 f}{\partial y^2} = 2\frac{x^2-y^2+z^2}{\left(x^2+y^2+z^2\right)^2}$ et $\frac{\partial^2 f}{\partial z^2} = 2\frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^2}$

$$\Delta f = 2\frac{-x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2} + 2\frac{x^2-y^2+z^2}{\left(x^2+y^2+z^2\right)^2} + 2\frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{2}{x^2+y^2+z^2} = \frac{2}{r^2}$$

4. $f(x,y,z) = e^{-xyz}$

$\frac{\partial f}{\partial x} = -yze^{-xyz}$ $\quad \frac{\partial^2 f}{\partial x^2} = y^2z^2e^{-xyz}$

$\frac{\partial f}{\partial y} = -xze^{-xyz}$ $\quad \frac{\partial^2 f}{\partial x^2} = x^2z^2e^{-xyz}$

$\frac{\partial f}{\partial z} = -xye^{-xyz}$ $\quad \frac{\partial^2 f}{\partial x^2} = x^2y^2e^{-xyz}$

$\overrightarrow{\nabla} f = -\left(yz\overrightarrow{i} + xz\overrightarrow{j} + xy\overrightarrow{k}\right)e^{-xyz}$

$\Delta f = \left(y^2z^2 + x^2z^2 + x^2y^2\right)e^{-xyz}$

5. $f(x,y,z) = e^{ax+by+cz}$

$\frac{\partial f}{\partial x} = ae^{ax+by+cz}$ $\quad \frac{\partial^2 f}{\partial x^2} = a^2e^{ax+by+cz}$

$\frac{\partial f}{\partial y} = be^{ax+by+cz}$ $\quad \frac{\partial^2 f}{\partial y^2} = b^2e^{ax+by+cz}$

$\frac{\partial f}{\partial z} = ce^{ax+by+cz}$ $\quad \frac{\partial^2 f}{\partial z^2} = c^2e^{ax+by+cz}$

$\overrightarrow{\nabla} f = \left(a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}\right)e^{ax+by+cz}$

$\Delta f = \left(a^2+b^2+c^2\right)e^{ax+by+cz}$

6. $f(x,y,z) = \left(x^2+y^2+z^2\right)^n$

$\frac{\partial f}{\partial x} = 2nx\left(x^2+y^2+z^2\right)^{n-1}$

$\frac{\partial f}{\partial y} = 2ny\left(x^2+y^2+z^2\right)^{n-1}$

$\frac{\partial f}{\partial z} = 2nz\left(x^2+y^2+z^2\right)^{n-1}$

$\overrightarrow{\nabla} f = 2n\left(x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}\right)\left(x^2+y^2+z^2\right)^{n-1}$

$\frac{\partial^2 f}{\partial x^2} = 2n\left(x^2+y^2+z^2\right)^{n-1} + 4n(n-1)x^2\left(x^2+y^2+z^2\right)^{n-2}$

$\frac{\partial^2 f}{\partial y^2} = 2n\left(x^2+y^2+z^2\right)^{n-1} + 4n(n-1)y^2\left(x^2+y^2+z^2\right)^{n-2}$

$\frac{\partial^2 f}{\partial z^2} = 2n\left(x^2+y^2+z^2\right)^{n-1} + 4n(n-1)z^2\left(x^2+y^2+z^2\right)^{n-2}$

$\Delta f = 6n\left(x^2+y^2+z^2\right)^{n-1} + 4n(n-1)\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{n-2}$

$= \left(6n + 4n(n-1)\right)\left(x^2+y^2+z^2\right)^{n-1}$

Exercice 2

Calculer les dérivées directionnelles de la fonction f suivant la direction $\overrightarrow{u}$ dans les cas suivantes :

1. $f(x,y) = x^2y + 4y^2, \quad \overrightarrow{u} = 2\overrightarrow{i} + \overrightarrow{j}$
2. $f(x,y) = \cos(2x - y), \quad \overrightarrow{u} = \overrightarrow{MN}, \text{ avec } M(\pi, 0) \quad N(2\pi, \pi)$
3. $f(x,y,z) = x^2z - 2xy + y^2 ; \quad \overrightarrow{u} = \overrightarrow{MN}, \text{ avec } M(-2,-1) \quad N(2,-3)$
4. $f(x,y,z) = y^2 + 2ye^{4xz} ; \quad \overrightarrow{u} = \overrightarrow{MN}, \text{ avec } M(0,-2,1) \quad N(-4,4,0)$
5. $f(x,y,z) = x^2 + y^2 + z^2 ; \quad \overrightarrow{u} = \overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$
6. $f(x,y,z) = xyz \quad \overrightarrow{u} = \overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$
7. $f(x,y,z) = \dfrac{1}{r} \quad \overrightarrow{u} = \overrightarrow{r}$

Solution 2 La dérivée directionnelle est définie par $\dfrac{\partial f}{\partial u} = \overrightarrow{\nabla} f \cdot \overrightarrow{e}$ avec $\overrightarrow{e} = \dfrac{\overrightarrow{u}}{u}$

1. $\overrightarrow{u} = 2\overrightarrow{i} + \overrightarrow{j}, \; u = \sqrt{4+1} = \sqrt{5} \Longrightarrow \overrightarrow{e} = \dfrac{2\overrightarrow{i} + \overrightarrow{j}}{\sqrt{5}}$

$\overrightarrow{\nabla} f = \dfrac{\partial f}{\partial x}\overrightarrow{i} + \dfrac{\partial f}{\partial y}\overrightarrow{j} + \dfrac{\partial f}{\partial z}\overrightarrow{k} = 2xy\overrightarrow{i} + (x^2 + 8y)\overrightarrow{j} + 0.\overrightarrow{k}$

$\dfrac{\partial f}{\partial u} = \overrightarrow{\nabla} f \cdot \overrightarrow{e} = \dfrac{1}{\sqrt{5}}\left(2xy\overrightarrow{i} + (x^2+8y)\overrightarrow{j}\right) \cdot \left(2\overrightarrow{i} + \overrightarrow{j}\right) = \dfrac{1}{\sqrt{5}}(4xy + x^2 + 8y)$

2. $\overrightarrow{\nabla} f = -2\sin(2x-y)\overrightarrow{i} + \sin(2x-y)\overrightarrow{j},$

$\overrightarrow{u} = \overrightarrow{MN} = \pi\overrightarrow{i} + \pi\overrightarrow{j}, u = \pi\sqrt{2} \Longrightarrow \overrightarrow{e} = \dfrac{1}{\sqrt{2}}\left(\overrightarrow{i} + \overrightarrow{j}\right)$

$\dfrac{\partial f}{\partial u} = \dfrac{1}{\sqrt{2}}(-2\sin(2x-y) + \sin(2x-y)) = -\dfrac{\sin(2x-y)}{\sqrt{2}}$

3. $\overrightarrow{\nabla} f = (2xz - 2y)\overrightarrow{i} + (-2x + 2y)\overrightarrow{j} + x^2\overrightarrow{k}$

$\overrightarrow{u} = \overrightarrow{MN} = 4\overrightarrow{i} - 2\overrightarrow{j} ; u = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

$\Longrightarrow \overrightarrow{e} = \dfrac{1}{2\sqrt{5}}\left(4\overrightarrow{i} - 2\overrightarrow{j}\right) = \dfrac{1}{\sqrt{5}}\left(2\overrightarrow{i} - \overrightarrow{j}\right)$

$\dfrac{\partial f}{\partial u} = \dfrac{1}{\sqrt{5}}(2(2xz - 2y) - (-2x + 2y)) = \dfrac{1}{\sqrt{5}}(2x - 6y + 4xz)$

4. $\overrightarrow{\nabla} f = 8yze^{4xz}\overrightarrow{i} + (2y + 2e^{4xz})\overrightarrow{j} + 8xye^{4xz}\overrightarrow{k}$

$\overrightarrow{u} = \overrightarrow{MN} = -4\overrightarrow{i} + 6\overrightarrow{j} - \overrightarrow{k}, u = \sqrt{16+36+1} = \sqrt{53}$

$\Longrightarrow \overrightarrow{e} = \dfrac{1}{\sqrt{53}}\left(-4\overrightarrow{i} + 6\overrightarrow{j} - \overrightarrow{k}\right)$

$\dfrac{\partial f}{\partial u} = \dfrac{1}{\sqrt{53}}\left(-32yze^{4xz} + 6(2y + 2e^{4xz}) - 8xye^{4xz}\right)$

$= \dfrac{1}{\sqrt{53}}\left(12y + (12 - 8xy - 32yz)e^{4xz}\right)$

5. $\overrightarrow{\nabla} f = 2x\overrightarrow{i} + 2y\overrightarrow{j} + 2z\overrightarrow{k}$

$\overrightarrow{e} = \dfrac{1}{\sqrt{3}}\left(\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}\right)$

$\dfrac{\partial f}{\partial u} = \dfrac{2}{\sqrt{3}}(x + y + z)$

6. $\overrightarrow{\nabla} f = yz\overrightarrow{i} + xz\overrightarrow{j} + xy\overrightarrow{k}$

$$\overrightarrow{e} = \frac{1}{\sqrt{3}}\left(\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}\right)$$
$$\frac{\partial f}{\partial u} = \frac{1}{\sqrt{3}}(yz + xz + xy)$$

7. $f = \frac{1}{r} = \left(x^2 + y^2 + z^2\right)^{-1/2}$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(2x)\left(x^2 + y^2 + z^2\right)^{-3/2} = -\frac{x}{\left(x^2 + y^2 + z^2\right)^{-3/2}} = -\frac{x}{r^3}$$

de même : $\frac{\partial f}{\partial y} = -\frac{y}{r^3}$ et $\frac{\partial f}{\partial z} = -\frac{z}{r^3}$

$$\overrightarrow{\nabla} f = -\frac{x}{r^3}\overrightarrow{i} - \frac{y}{r^3}\overrightarrow{j} - \frac{z}{r^3}\overrightarrow{k} = -\frac{\overrightarrow{r}}{r^3}$$
$$\overrightarrow{e} = \frac{\overrightarrow{u}}{u} = \frac{\overrightarrow{r}}{r}$$
$$\frac{\partial f}{\partial u} = \overrightarrow{\nabla} f \cdot \overrightarrow{e} = -\frac{\overrightarrow{r}}{r^3} \cdot \frac{\overrightarrow{r}}{r} = -\frac{r^2}{r^4} = -\frac{1}{r^2}$$

Exercice 3

Déterminer la normale unitaire en un point M,des surfaces suivantes :

1. plan : $ax + by + cz + d = 0$
2. Sphère (O, R)
3. Ellipsoïde $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
4. $f(x, y, z) = \ln(x + y + z)$
5. $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$
6. $z = 2 - x^4 - y^4$

Solution 3 : La normale unitaire sur la surface définie par $f(x, y, z) = 0$ est

$$\overrightarrow{n} = \frac{\overrightarrow{\nabla} f}{\left|\overrightarrow{\nabla} f\right|}$$

1. $f = ax + by + cz + d$

$$\overrightarrow{\nabla} f = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k} \Longrightarrow \left|\overrightarrow{\nabla} f\right| = \sqrt{a^2 + b^2 + c^2}$$
$$\overrightarrow{n} = \frac{a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}}{\sqrt{a^2 + b^2 + c^2}}$$

2. Equation de sphère (O, R) : $f(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$

$$\overrightarrow{\nabla} f = 2x\overrightarrow{i} + 2y\overrightarrow{j} + 2z\overrightarrow{k} \Longrightarrow \left|\overrightarrow{\nabla} f\right| = 2\sqrt{x^2 + y^2 + z^2} = 2R$$
$$\overrightarrow{n} = \frac{1}{R}\left(x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}\right) = \frac{\overrightarrow{r}}{R}$$

3. $f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

$$\overrightarrow{\nabla} f = \frac{2x}{a^2}\overrightarrow{i} + \frac{2y}{b^2}\overrightarrow{j} + \frac{2z}{c^2}\overrightarrow{k}$$

$$\left|\vec{\nabla} f\right| = 2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

$$\vec{n} = \frac{\frac{x}{a^2}\vec{i}+\frac{y}{b^2}\vec{j}+\frac{z}{c^2}\vec{k}}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}$$

4. $\vec{\nabla} f = \dfrac{\vec{i}+\vec{j}+\vec{k}}{x+y+z} \Longrightarrow \left|\vec{\nabla} f\right| = \dfrac{\sqrt{3}}{x+y+z}$

$$\vec{n} = \frac{1}{\sqrt{3}}\left(\vec{i}+\vec{j}+\vec{k}\right)$$

5. $\dfrac{\partial f}{\partial x} = \dfrac{1}{2}(-2x)\left(1-x^2-y^2-z^2\right)^{-1/2} = -\dfrac{x}{\sqrt{1-x^2-y^2-z^2}}$

de même $\dfrac{\partial f}{\partial y} = -\dfrac{y}{\sqrt{1-x^2-y^2-z^2}}$ et $\dfrac{\partial f}{\partial z} = -\dfrac{z}{\sqrt{1-x^2-y^2-z^2}}$

$\vec{\nabla} f = -\dfrac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{1-x^2-y^2-z^2}} \Longrightarrow \left|\vec{\nabla} f\right| = \dfrac{\sqrt{x^2+y^2+z^2}}{\sqrt{1-x^2-y^2-z^2}}$

$$\vec{n} = -\frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}}$$

6. $f = 2 - x^4 - y^4 - z$

$\vec{\nabla} f = -4x^3\vec{i} - 4y^3\vec{j} - \vec{k} \Longrightarrow \left|\vec{\nabla} f\right| = \sqrt{16x^6+16y^6+1}$

$$\vec{n} = \frac{-4x^3\vec{i}-4y^3\vec{j}-\vec{k}}{\sqrt{16x^6+16y^6+1}}$$

Exercice 4

Soit $\vec{H} = (x+3y)\,\vec{i} + (y-2z)\,\vec{j} + (x+az)\,\vec{k}$ et $\varphi(x,y,z) = xy^3 \sin xz$

1. Déterminer la constante a pour que div $\vec{H} = 0$, et calculer dans ce cas $\vec{\nabla} \times \vec{H}$.
2. Calculer $\vec{\nabla}\varphi$ et $\Delta\varphi$
3. Calculer $\vec{\nabla} \cdot \left(\varphi\vec{H}\right)$

Solution 4 :

1. $div\,\vec{H} = \vec{\nabla}\cdot\vec{H} = \dfrac{\partial}{\partial x}(x+3y) + \dfrac{\partial}{\partial y}(y-2z) + \dfrac{\partial}{\partial z}(x+az) = 1+1+a = 2+a$

$div\,\vec{H} = 0 \Longleftrightarrow a+2=0 \Longrightarrow a=-2$ soit donc

$$\vec{H} = (x+3y)\,\vec{i} + (y-2z)\,\vec{j} + (x-2z)\,\vec{k} \qquad (1)$$

$$\vec{\nabla}\times\vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+3y & y-2z & x-2z \end{vmatrix} = 2\,\vec{i} - \vec{j} - 3\,\vec{k}$$

2. $\varphi(x,y,z) = xy^3 \sin xz$

$$\vec{\nabla}\varphi = \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} + \frac{\partial \varphi}{\partial z}\vec{k}$$
$$= \left(y^3 \sin xz + xy^3 z \cos xz\right)\vec{i} + 3xy^2 \sin xz\,\vec{j} + x^2y^3 \cos xz\,\vec{k}$$
$$\Delta\varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$
$$= \left(zy^3 \cos xz + y^3 z \cos xz - xy^3 z^2 \sin xz\right) + 6xy \sin xz - x^3y^3 \sin xz$$
$$= 6xy \sin xz - x^3y^3 \sin xz + 2y^3 z \cos xz - xy^3 z^2 \sin xz$$

$$\Delta\varphi = \left(6xy - x^3y^3 - xy^3z^2\right)\sin xz + 2y^3 z \cos xz$$

3. $\vec{\nabla}\cdot\left(\varphi\vec{H}\right) = \varphi\vec{\nabla}\cdot\vec{H} + \vec{\nabla}\varphi\cdot\vec{H}$

On a pour $a = 2$, $\vec{\nabla}\cdot\vec{H} = 0 \Longrightarrow \vec{\nabla}\cdot\left(\varphi\vec{H}\right) = \vec{\nabla}\varphi\cdot\vec{H}$

$$= (x+3y)\left(y^3 \sin xz + xy^3 z \cos xz\right) + (y-2z)\times 3xy^2 \sin xz + (x-2z)\,x^2y^3 \cos xz$$
$$= 3y^4 \sin xz + x^3y^3 \cos xz + 4xy^3 \sin xz - x^2y^3 z \cos xz + 3xy^4 z \cos xz - 6xy^2 z \sin xz$$

$$\vec{\nabla}\cdot\left(\varphi\vec{H}\right) = \left(3y^4 + 4xy^3 - 6xy^2z\right)\sin xz + \left(x^3y^3 - x^2y^3z + 3xy^4z\right)\cos xz$$

Exercice 5

Soit $\vec{H}$ un champ vectoriel défini dans l'espace rapporté au système $\left(O, \vec{i}, \vec{j}, \vec{k}\right)$

1. Calculer les produits vectoreils : $\vec{i}\times\vec{j}$, $\vec{i}\times\vec{k}$ et $\vec{j}\times\vec{k}$
2. Montrer que
$$\vec{\nabla}\times\vec{H} = \vec{i}\times\frac{\partial\vec{H}}{\partial x} + \vec{j}\times\frac{\partial\vec{H}}{\partial y} + \vec{k}\times\frac{\partial\vec{H}}{\partial z}$$
3. Application : $\vec{H} = \left(x^2+y^2\right)\vec{i} + \left(y^2+z^2\right)\vec{j} + \left(x^2+z^2\right)\vec{k}$

Solution 5 :

1. $\left(O, \vec{i}, \vec{j}, \vec{k}\right)$ est un système direct, alors

$$\vec{i}\times\vec{j} = \vec{k},\ \vec{i}\times\vec{k} = -\vec{j}\ et\ \vec{j}\times\vec{k} = \vec{i}$$

de plus
$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$$

2. Soit $\vec{H} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$

$$\vec{i}\times\frac{\partial\vec{H}}{\partial x} = \vec{i}\times\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial Q}{\partial x}\vec{j} + \frac{\partial R}{\partial x}\vec{k}\right)$$
$$= \frac{\partial P}{\partial x}\underbrace{\vec{i}\times\vec{i}}_{=\vec{0}} + \frac{\partial Q}{\partial x}\underbrace{\vec{i}\times\vec{j}}_{=\vec{k}} + \frac{\partial R}{\partial x}\underbrace{\vec{i}\times\vec{k}}_{=-\vec{j}} = \frac{\partial Q}{\partial x}\vec{k} - \frac{\partial R}{\partial x}\vec{j}$$
$$\vec{j}\times\frac{\partial\vec{H}}{\partial y} = \vec{j}\times\left(\frac{\partial P}{\partial y}\vec{i} + \frac{\partial Q}{\partial y}\vec{j} + \frac{\partial R}{\partial y}\vec{k}\right) = -\frac{\partial P}{\partial y}\vec{k} + \frac{\partial R}{\partial y}\vec{i}$$

$$\vec{k} \times \frac{\partial \vec{H}}{\partial z} = \vec{k} \times \left(\frac{\partial P}{\partial z} \vec{i} + \frac{\partial Q}{\partial z} \vec{j} + \frac{\partial R}{\partial z} \vec{k} \right) = \frac{\partial P}{\partial z} \vec{j} - \frac{\partial Q}{\partial z} \vec{i}$$

$$\vec{i} \times \frac{\partial \vec{H}}{\partial x} + \vec{j} \times \frac{\partial \vec{H}}{\partial y} + \vec{k} \times \frac{\partial \vec{H}}{\partial z} = \frac{\partial Q}{\partial x} \vec{k} - \frac{\partial R}{\partial x} \vec{j} - \frac{\partial P}{\partial y} \vec{k} + \frac{\partial R}{\partial y} \vec{i} + \frac{\partial P}{\partial z} \vec{j} - \frac{\partial Q}{\partial z} \vec{i}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} = \vec{\nabla} \times \vec{H}$$

3. Application :

$$\vec{\nabla} \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & y^2 + z^2 & x^2 + z^2 \end{vmatrix} = -2z \, \vec{i} - 2x \, \vec{j} - 2y \, \vec{k}$$

$$\vec{i} \times \frac{\partial \vec{H}}{\partial x} = \vec{i} \times \left(2x \, \vec{i} + 2x \, \vec{k} \right) = -2x \, \vec{j}$$

$$\vec{j} \times \frac{\partial \vec{H}}{\partial y} = \vec{j} \times \left(2y \, \vec{i} + 2y \, \vec{j} \right) = -2y \, \vec{k}$$

$$\vec{k} \times \frac{\partial \vec{H}}{\partial z} = \vec{k} \times \left(2z \, \vec{j} + 2y \, \vec{k} \right) = -2z \, \vec{i}$$

$$\vec{i} \times \frac{\partial \vec{H}}{\partial x} + \vec{j} \times \frac{\partial \vec{H}}{\partial y} + \vec{k} \times \frac{\partial \vec{H}}{\partial z} = -2z \, \vec{i} - 2x \, \vec{j} - 2y \, \vec{k} = \vec{\nabla} \times \vec{H}$$

Pour calculer rapidement le rotationnel de $\vec{H}$ on peut utiliser la notation suivante :

$$\vec{\nabla} \times \vec{H} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \searrow \\ \frac{\partial}{\partial z} \nearrow \end{pmatrix} \begin{pmatrix} P \\ \nearrow Q \\ \searrow R \end{pmatrix} = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \qquad \begin{pmatrix} \frac{\partial}{\partial x} \searrow \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \nearrow \end{pmatrix} \begin{pmatrix} \nearrow P \\ Q \\ \searrow R \end{pmatrix} = -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right)$$

$$et \begin{pmatrix} \frac{\partial}{\partial x} \searrow \\ \frac{\partial}{\partial y} \nearrow \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \nearrow P \\ \searrow Q \\ R \end{pmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

Exercice 6

oit $\vec{v}\,(M)$ le champ de vitesses des points d'un solide qui tourne, autour d'un axe fixe (D), avec la vitesse de rotation $\vec{\omega}$ constante. Dans ce cas on a $\vec{v} = \vec{\omega} \times \vec{r}$, à condition que l'axe de rotation passe par l'origine de coordonnées de $\vec{r}$ des points M de solide.

Si on considère le cas où $\vec{\omega} = \omega \, \vec{k}$ et $\vec{r} = x \, \vec{i} + y \, \vec{j}$

1. Calculer les composantes de $\vec{v}$

2. Calculer div $\overrightarrow{v}$ et montrer que $\overrightarrow{\omega} = \frac{1}{2}\overrightarrow{\nabla} \times \overrightarrow{v}$

Solution 6 :

1. $$\overrightarrow{v} = \overrightarrow{\omega} \times \overrightarrow{r} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = -\omega y \overrightarrow{i} + \omega x \overrightarrow{j} = v_x \overrightarrow{i} + v_y \overrightarrow{j}$$

2. $$\text{div}\,\overrightarrow{v} = \overrightarrow{\nabla} \cdot \overrightarrow{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\overrightarrow{\nabla} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = (0-0)\,\overrightarrow{i} - (0-0)\,\overrightarrow{j} + (\omega + \omega)\,\overrightarrow{k}$$

$$= 2\omega \overrightarrow{k} = 2\overrightarrow{\omega} \Longleftrightarrow \overrightarrow{\omega} = \frac{1}{2}\overrightarrow{\nabla} \times \overrightarrow{v}$$

Exercice 7

On considère les champs scalaires :

$$\begin{aligned} f(x, y, z, t) &= Ae^{j(\omega t - ax - by - cz)} \\ g(x, y, z, t) &= \sin n\pi (x + y + z) \cos n\pi\omega t \end{aligned}$$

où A, ω, a, b, c et n sont des constantes données, x, y, z, t sont des variables réelles.
On pose $k = \sqrt{a^2 + b^2 + c^2}$ et $v = \frac{\omega}{k}$. Montrer que :

1. $\Delta f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$
2. $\Delta g - \frac{3}{\omega^2}\frac{\partial^2 g}{\partial t^2} = 0$

Solution 7 :

1. $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(A \exp\left(j\left(\omega t - ax - by - cz\right)\right)\right)$$

$$= -jaA \exp\left(j\left(\omega t - ax - by - cz\right)\right) = -jaf$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-jaf\right) = -ja\frac{\partial f}{\partial x} = \left(-ja\right)^2 f = -a^2 f$$

de même on trouve :

$$\frac{\partial^2 f}{\partial y^2} = -b^2 f \qquad \frac{\partial^2 f}{\partial z^2} = -c^2 f$$

donc $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -\left(a^2 + b^2 + c^2\right) f = -k^2 f$

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 f$$

On a $f = -\frac{1}{k^2}\Delta f = -\frac{1}{\omega^2}\frac{\partial^2 f}{\partial t^2} \Longrightarrow \Delta f = \frac{k^2}{\omega^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$
d'où finalement :

$$\Delta f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

2. $g(x,y,z,t) = \sin n\pi(x+y+z)\cos n\pi\omega t$
$\frac{\partial^2}{\partial t^2}(\sin n\pi(x+y+z)\cos n\pi\omega t) = -\pi^2\omega^2 n^2 \sin n\pi(x+y+z)\cos n\pi\omega t$
$= -\pi^2 n^2\omega^2 g$
$\frac{\partial^2}{\partial x^2}(\sin n\pi(x+y+z)\cos n\pi\omega t) = -\pi^2 n^2 \sin n\pi(x+y+z)\cos n\pi\omega t$
$= -\pi^2 n^2 g$
alors : $\Delta g = -3\pi^2 n^2 g$
Remarquons que : $-\pi^2 n^2 g = \frac{\Delta g}{3} = \frac{1}{\omega^2}\frac{\partial^2 g}{\partial t^2}$ donc :

$$\Delta g - \frac{3}{\omega^2}\frac{\partial^2 g}{\partial t^2} = 0$$

Exercice 8

Une charge électrique q est soumise, du côté d'une charge fixe Q à la force de Coulombe :

$$\overrightarrow{F} = \frac{qQ}{4\pi\varepsilon}\frac{\overrightarrow{r}}{r^3}$$

1. Calculer :
   (a) div $\overrightarrow{F}$
   (b) $\overrightarrow{\nabla} \times \overrightarrow{F}$
2. Montrer que $\overrightarrow{F}$ dérive d'un potentiel scalaire $\varphi = \varphi(x,y,z)$ et déterminer l'expression de φ si $\varphi(\infty) = 0$

Solution 8 :

1. $\overrightarrow{F} = \frac{qQ}{4\pi\varepsilon}\frac{\overrightarrow{r}}{r^3} = \frac{qQ}{4\pi\varepsilon}\frac{x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}}{(x^2+y^2+z^2)^{3/2}}$.

   (a) div $\overrightarrow{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

$$\frac{\partial F_x}{\partial x} = \frac{qQ}{4\pi\varepsilon}\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{Qq}{4\pi\varepsilon}\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$
$$= \frac{Qq}{4\pi\varepsilon}\frac{-3x^2+x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} = \frac{Qq}{4\pi\varepsilon}\frac{-3x^2+r^2}{r^5}$$
$$\frac{\partial F_y}{\partial y} = \frac{qQ}{4\pi\varepsilon}\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{Qq}{4\pi\varepsilon}\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} = \frac{Qq}{4\pi\varepsilon}\frac{-3y^2+r^2}{r^5}$$

$$\frac{\partial F_z}{\partial z} = \frac{qQ}{4\pi\varepsilon}\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{Qq}{4\pi\varepsilon}\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} = \frac{Qq}{4\pi\varepsilon}\frac{-3z^2+r^2}{r^5}$$

$$\text{div}\,\vec{F} = \frac{Qq}{4\pi\varepsilon}\left(\frac{-3x^2+r^2}{r^5}+\frac{-3y^2+r^2}{r^5}+\frac{-3z^2+r^2}{r^5}\right)$$

$$= \frac{Qq}{4\pi\varepsilon}\frac{-3\left(x^2+y^2+z^2\right)+r^2}{r^5} = 0$$

(b) $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{qQ}{4\pi\varepsilon}\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) = -\frac{3Qq}{4\pi\varepsilon}\frac{xy}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{qQ}{4\pi\varepsilon}\frac{y}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) = -\frac{3Qq}{4\pi\varepsilon}\frac{xy}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\Longrightarrow \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

de même on trouve $\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$ et $\frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z} \Longrightarrow \vec{\nabla}\times\vec{F} = \vec{0}$

2. Puisque $\vec{\nabla}\times\vec{F} = \vec{0}$ donc $\vec{F}$ dérive d'un potentiel scalaire, c'est-à-dire il existe un champ scalaire $\varphi = \varphi(x,y,z)$ tel que $\vec{F} = \vec{\nabla}\varphi$

(a) $$F_x = \frac{qQ}{4\pi\varepsilon}\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial\varphi}{\partial x}$$

$$\Longrightarrow \varphi = \int \frac{qQ}{4\pi\varepsilon}\frac{xdx}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = -\frac{qQ}{4\pi\varepsilon}\frac{1}{\sqrt{x^2+y^2+z^2}} + f(y,z)$$

$$F_y = \frac{qQ}{4\pi\varepsilon}\frac{y}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial\varphi}{\partial y}$$

$$= \frac{\partial}{\partial y}\left(-\frac{qQ}{4\pi\varepsilon}\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial f}{\partial y} = \frac{Qq}{4\pi\varepsilon}\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{\partial f}{\partial y}$$

$$\Longrightarrow \frac{\partial f}{\partial y} = 0 \Longrightarrow f = f(z)$$

$$\varphi = -\frac{qQ}{4\pi\varepsilon}\frac{1}{\sqrt{x^2+y^2+z^2}} + f(z)$$

$$F_y = \frac{qQ}{4\pi\varepsilon}\frac{z}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial\varphi}{\partial z}$$

$$= \frac{\partial}{\partial z}\left(-\frac{qQ}{4\pi\varepsilon}\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial f}{\partial z} = \frac{Qq}{4\pi\varepsilon}\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{\partial f}{\partial z}$$

$$\Longrightarrow \frac{\partial f}{\partial z} = 0 \Longrightarrow f = Const = C$$

$$\varphi = -\frac{qQ}{4\pi\varepsilon}\frac{1}{\sqrt{x^2+y^2+z^2}} + C = -\frac{qQ}{4\pi\varepsilon r} + C$$

On peut calculer φ directemnt en fonction de r; en effet :
Soit $\overrightarrow{dr} = dx\,\overrightarrow{i} + dy\,\overrightarrow{j} + dz$

$\overrightarrow{F} = \overrightarrow{\nabla}\varphi \Longrightarrow \overrightarrow{F}\cdot\overrightarrow{dr} = \overrightarrow{\nabla}\varphi\cdot\overrightarrow{dr} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi$

donc $d\varphi = \overrightarrow{F}\cdot\overrightarrow{dr} = \frac{qQ}{4\pi\varepsilon}\frac{\overrightarrow{r}}{r^3}\cdot\overrightarrow{dr} = \frac{qQ}{4\pi\varepsilon}\frac{rdr}{r^3} = \frac{qQ}{4\pi\varepsilon}\frac{dr}{r^2}$

$\varphi = \int \frac{qQ}{4\pi\varepsilon}\frac{dr}{r^2} = -\frac{1}{4\pi\varepsilon}\frac{qQ}{r} + C$

Exercice 9

On considère le champ vectoriel :

$$\overrightarrow{F}(M) = \frac{\overrightarrow{r}}{r^2+1}$$

1. Calculer $\overrightarrow{\nabla}\cdot\overrightarrow{F}$ et $\overrightarrow{\nabla}\times\overrightarrow{F}$
2. $\overrightarrow{F}$ est-il un champ conservatif ?. Justifier la réponse.
3. Déterminer le potentiel scalaire $\varphi = \varphi(x,y,z)$ de $\overrightarrow{F}$ tel que $\varphi(0) = 0$.

Solution 9 : $\overrightarrow{F}(M) = \frac{\overrightarrow{r}}{r^2+1} = \frac{x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}}{x^2+y^2+z^2+1} = P\,\overrightarrow{i} + Q\,\overrightarrow{j} + R\,\overrightarrow{k}$

1. $\overrightarrow{\nabla}\cdot\overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2+1}\right) = \frac{-x^2+y^2+z^2+1}{(x^2+y^2+z^2+1)^2} = \frac{r^2+1-2x^2}{(r^2+1)^2}$$

de même $\frac{\partial Q}{\partial y} = \frac{r^2+1-2y^2}{(r^2+1)^2}$

et $\frac{\partial R}{\partial z} = \frac{r^2+1-2z^2}{(r^2+1)^2}$

$$\overrightarrow{\nabla}\cdot\overrightarrow{F} = \frac{r^2+1-2x^2}{(r^2+1)^2} + \frac{r^2+1-2y^2}{(r^2+1)^2} + \frac{r^2+1-2z^2}{(r^2+1)^2} = \frac{r^2+3}{(r^2+1)^2}$$

– On peut utiliser aussi la formule : $\text{div}\left(u\overrightarrow{V}\right) = \overrightarrow{\text{grad}}u\cdot\overrightarrow{V} + u\,\text{div}\,\overrightarrow{V}$

$$\overrightarrow{\nabla}\cdot\frac{\overrightarrow{r}}{r^2+1} = \overrightarrow{\nabla}\left(\frac{1}{r^2+1}\right)\cdot\overrightarrow{r} + \frac{\overrightarrow{\nabla}\cdot\overrightarrow{r}}{r^2+1}$$

$$= \frac{-2x\,\overrightarrow{i} - 2y\,\overrightarrow{j} - 2z\,\overrightarrow{k}}{(r^2+1)^2}\cdot\overrightarrow{r} + \frac{3}{r^2+1}$$

$$= \frac{-2r^2}{(r^2+1)^2} + \frac{3}{r^2+1} = \frac{r^2+3}{(r^2+1)^2}$$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{x^2+y^2+z^2+1}\right) = \frac{-2yz}{(x^2+y^2+z^2+1)^2} = \frac{\partial Q}{\partial z}$$

de même on trouve

$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = \frac{-2xz}{(x^2+y^2+z^2+1)^2}$$

et $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{-2xy}{(x^2+y^2+z^2+1)^2}$

Donc $\vec{\nabla} \times \vec{F} = \vec{0}$

2. $\vec{\nabla} \times \vec{F} = \vec{0}$ donc $\vec{F}$ est un champ conservatif

3. $\exists\, \varphi = \varphi(x,y,z) \,/\, \vec{F} = \overrightarrow{\nabla\varphi}$

$$\Longrightarrow \vec{F} \cdot \vec{dr} = \overrightarrow{\nabla\varphi} \cdot \vec{dr} = d\varphi$$

$$\varphi = \int d\varphi = \int \vec{F} \cdot \vec{dr} = \int \frac{\vec{r} \cdot \vec{dr}}{r^2+1}$$

$$= \int \frac{rdr}{r^2+1} = \frac{1}{2}\ln\left(r^2+1\right) + C$$

$$\varphi(0) = \frac{1}{2}\ln(0+1) + C = 0 \Longrightarrow C = 0$$

$$\varphi(x,y,z) = \ln\sqrt{x^2+y^2+z^2+1}$$

Exercice 10

On considère le champ vectoriel

$$\vec{H} = \vec{r}\exp\left(-r^2\right)$$

1. Calculer $\vec{\nabla} \cdot \vec{H}$ et $\vec{\nabla} \times \vec{H}$.
2. Montrer que $\vec{H}$ est un champ de gradient.
3. Déterminer le champ scalaire $f(x,y,z)$ tel que $\vec{H} = \overrightarrow{\nabla f}$ et $f(\infty,\infty,\infty) = 0$

Solution 10 $\vec{H} = \vec{r}\exp\left(-r^2\right) = P\vec{i} + Q\vec{j} + R\vec{k}$

1. $\vec{\nabla} \cdot \vec{H} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$

$$P = xe^{-x^2-y^2-z^2} \Longrightarrow \frac{\partial P}{\partial x} = e^{-x^2-y^2-z^2} - 2x^2e^{-x^2-y^2-z^2} = \left(1-2x^2\right)e^{-x^2-y^2-z^2}$$

$$Q = ye^{-x^2-y^2-z^2} \Longrightarrow \frac{\partial Q}{\partial y} = e^{-x^2-y^2-z^2} - 2y^2e^{-x^2-y^2-z^2} = \left(1-2y^2\right)e^{-x^2-y^2-z^2}$$

$R = ze^{-x^2-y^2-z^2} \Longrightarrow \dfrac{\partial R}{\partial z} = \left(1-2z^2\right)e^{-x^2-y^2-z^2}$ alors

$$\vec{\nabla} \cdot \vec{H} = \left(3-2r^2\right)e^{-r^2}$$

$$\vec{\nabla} \times \vec{H} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\frac{\partial R}{\partial y} = -2yze^{-x^2-y^2-z^2} \qquad \frac{\partial Q}{\partial z} = -2yze^{-x^2-y^2-z^2} \Longrightarrow \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0$$

de même on trouve : $\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ par suite

$$\vec{\nabla} \times \vec{H} = \vec{0}$$

2. $\vec{\nabla} \times \vec{H} = \vec{0} \Longrightarrow \vec{H}$ est un champ de gradient.

3. $\vec{H}$ est un champ de gradient, c-à-d il existe une fonction $f(x,y,z)$; $\vec{H} = \vec{\nabla} f$

C'est-à-dire : $\vec{H} \cdot \vec{dr} = df$

$df = \vec{H} \cdot \vec{dr} = \vec{r} \exp(-r^2) \cdot \vec{dr} = r \exp(-r^2)\, dr$

$f = \int r \exp(-r^2)\, dr = -\frac{1}{2}e^{-r^2} + C$

$f(\infty) = C = 0$ d'où finalement :

$$f(x,y,z) = -\frac{1}{2}e^{-r^2} = -\frac{1}{2}e^{-x^2-y^2-z^2}$$

Exercice 11

On considère le champ vectoriel :

$$\vec{W} = \left(x^\alpha + xy^2 + xz^2\right)\vec{i} + \left(x^2y + y^\beta + yz^2\right)\vec{j} + \left(x^2z + y^2z + z^\gamma\right)\vec{k}$$

1. Calculer $\overrightarrow{rotW}$.

2. Pour quelles valeurs de α, β, et γ le champ $\vec{W}$ est un champ de gradient ?

3. Soit $\alpha = \beta = \gamma = 3$.
Exprimer $\vec{W}$ à l'aide du rayon vecteur $\vec{r} = \overrightarrow{OM}$. et $r = \|\vec{r}\|$.

4. Déterminer dans les conditions 3) le champ scalaire U ; $\vec{W} = \overrightarrow{\text{grad}\, U}$.

Solution 11 :

1. $$\vec{\nabla} \times \vec{W} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^\alpha + xy^2 + xz^2 & x^2y + y^\beta + yz^2 & x^2z + y^2z + z^\gamma \end{vmatrix}$$

$= (2yz - 2yz)\vec{i} - (2xz - 2xz)\vec{j} + (2xy - 2xy)\vec{k} = \vec{0}$

2. On a $\vec{\nabla} \times \vec{W} = \vec{0}$ indépendament des valeurs de α, β, et γ alors $\vec{W}$ est un champ de gradient $\forall\, \alpha, \beta, \gamma \in \mathbb{R}$.

3. $\alpha = \beta = \gamma = 3 \Longrightarrow$

$\vec{W} = (x^3 + xy^2 + xz^2)\vec{i} + (x^2y + y^3 + yz^2)\vec{j} + (x^2z + y^2z + z^3)\vec{k}$

$= x(x^2 + y^2 + z^2)\vec{i} + y(x^2 + y^2 + z^2)\vec{j} + z(x^2 + y^2 + z^2)\vec{k}$

$= (x^2 + y^2 + z^2)\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = r^2\vec{r}$

4. $\overrightarrow{W} = \overrightarrow{\text{grad}\, U} = \frac{\partial U}{\partial x}\overrightarrow{i} + \frac{\partial U}{\partial y}\overrightarrow{j} + \frac{\partial U}{\partial z}\overrightarrow{k}$

$$\frac{\partial U}{\partial x} = x^3 + xy^2 + xz^2$$

$$\Longrightarrow U(x,y,z) = \int (x^3 + xy^2 + xz^2)\, dx = \frac{1}{4}x^2(x^2 + 2y^2 + 2z^2) + f(y,z)$$

$$\frac{\partial U}{\partial y} = x^2 y + \frac{\partial f}{\partial y} = x^2 y + y^3 + z^2 y \Longrightarrow \frac{\partial f}{\partial y} = y^3 + z^2 y$$

$$\Longrightarrow f(y,z) = \int (y^3 + z^2 y)\, dy = \frac{1}{4}y^4 + \frac{1}{2}y^2 z^2 + g(z)$$

$$U(x,y,z) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}x^2z^2 + \frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + g(z)$$

$$\frac{\partial U}{\partial z} = x^2 z + y^2 z + g' = x^2 z + y^2 z + z^3$$

$$\Longrightarrow g' = z^3 \Longrightarrow g(z) = \frac{z^4}{4} + C$$

$$U(x,y,z) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}x^2z^2 + \frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + \frac{1}{4}z^4 + C$$

$$= \frac{1}{4}(x^4 + 2x^2y^2 + 2x^2z^2 + y^4 + 2y^2z^2 + z^4) + C = \frac{r^4}{4} + C$$

$$\overrightarrow{W} = r^2\overrightarrow{r} \Longrightarrow dU = \overrightarrow{W}\cdot\overrightarrow{dr} = r^3 dr \Longrightarrow U = \frac{r^4}{4} + C$$

Exercice 12

On considère le champ de vecteurs :

$$\overrightarrow{q} = (6xy + \beta z^3)\,\overrightarrow{i} + (3x^2 - z)\,\overrightarrow{j} + (3xz^2 - y + z\sin z)\,\overrightarrow{k}$$

1. Calculer div $\overrightarrow{q}$ et $\overrightarrow{rot\, q}$
2. Pour quelle valeur de β, $\overrightarrow{q}$ est un champ de gradient ?
3. Avec la valeur de β ainsi trouvée, déterminer le champ F tel que $\overrightarrow{q} = \overrightarrow{\nabla} F$ et $F(0,0,0) = 3$

Solution 12 :

1. $\overrightarrow{\nabla}\cdot\overrightarrow{q} = div\,\overrightarrow{q} = \frac{\partial}{\partial x}(6xy + z^3) + \frac{\partial}{\partial y}(3x^2 - z) + \frac{\partial}{\partial z}(3xz^2 - y + z\sin z)$

$$= 6y + 6xz + \sin z + z\cos z$$

$$\nabla \times \overrightarrow{q} = (3\beta z^2 - 3z^2)\,\overrightarrow{j}$$

2. $\overrightarrow{q}$ est un champ de gradient c.à.d. il existe $F(M)$ tel que

$$\overrightarrow{q} = \overrightarrow{\text{grad} F} \Longleftrightarrow \overrightarrow{\nabla}\times\overrightarrow{q} = \overrightarrow{0}$$

$$\nabla\times\overrightarrow{q} = (3\beta z^2 - 3z^2)\,\overrightarrow{j} \Rightarrow 3\beta z^2 - 3z^2 = 0 \Rightarrow \beta = 1$$

Pour $\beta = 1$:

$$\overrightarrow{q} = (6xy + z^3)\,\overrightarrow{i} + (3x^2 - z)\,\overrightarrow{j} + (3xz^2 - y + z\sin z)\,\overrightarrow{k}$$

3. $\overrightarrow{q} = \overrightarrow{\nabla} F = \dfrac{\partial F}{\partial x}\overrightarrow{i} + \dfrac{\partial F}{\partial y}\overrightarrow{j} + \dfrac{\partial F}{\partial z}\overrightarrow{k}$

$$\Rightarrow \begin{cases} F'_x = 6xy + z^3 \\ F'_y = 3x^2 - z \\ F'_z = 3xz^2 - y + z\sin z \end{cases}$$

$F'_x = 6xy + z^3 \Rightarrow F(x,y,z) = \int \left(6xy + z^3\right) dx = 3x^2y + z^3x + f(y,z)$

$F'_y = 3x^2 - z = 3x^2 + \dfrac{\partial f}{\partial y} \Rightarrow \dfrac{\partial f}{\partial y} = -z \Rightarrow f(y,z) = -yz + g(z)$

donc $F = 3x^2y + z^3x - yz + g(z)$

$F'_z = 3xz^2 - y + z\sin z = 3z^2x - y + g' \Rightarrow g' = z\sin z$

$\Rightarrow g = \int z\sin z dz = \sin z - z\cos z + C$

et $F = 3x^2y + z^3x - yz + \sin z - z\cos z + C$

$F(0,0,0) = C = 3$

$$F = 3x^2y + z^3x - yz + \sin z - z\cos z + 3$$

Exercice 13

Soit $u(M)$ un champ scalaire défini et continu dans un domaine D, et $\overrightarrow{V} = \overrightarrow{V}(M)$ un champ vectoriel.

1. Montrer que, si $f(u)$ est une fonction dérivable de u alors $\overrightarrow{\nabla} f = \dfrac{df}{du}\overrightarrow{\nabla} u$
2. Démontrer la formule : $\text{div}\left(u\overrightarrow{V}\right) = \overrightarrow{\text{grad}}\, u \cdot \overrightarrow{V} + u \cdot div\overrightarrow{V}$.
3. Démontrer que $\forall n \in \mathbb{Q}$; $\overrightarrow{\nabla} r^n = nr^{n-2}\overrightarrow{r}$
4. Déduire : $\overrightarrow{\nabla} r$, $\overrightarrow{\nabla}\left(\dfrac{1}{r}\right)$, $\overrightarrow{\nabla}\left(\dfrac{1}{r^2}\right)$, $\Delta\left(\dfrac{1}{r}\right)$
5. Soit $\overrightarrow{H} = \overrightarrow{r} f(r)$. Déterminer $f(r)$ telle que div $\overrightarrow{H} = 0$ et $f(1) = 1$

Solution 13 :

1. $\overrightarrow{\nabla} f = \dfrac{\partial f}{\partial x}\overrightarrow{i} + \dfrac{\partial f}{\partial y}\overrightarrow{j} + \dfrac{\partial f}{\partial z}\overrightarrow{k} = \dfrac{df}{du}\dfrac{\partial u}{\partial x}\overrightarrow{i} + \dfrac{df}{du}\dfrac{\partial u}{\partial y}\overrightarrow{j} + \dfrac{df}{du}\dfrac{\partial u}{\partial z}\overrightarrow{k}$

$= \dfrac{df}{du}\left(\dfrac{\partial u}{\partial x}\overrightarrow{i} + \dfrac{\partial u}{\partial y}\overrightarrow{j} + \dfrac{\partial u}{\partial z}\overrightarrow{k}\right) = \dfrac{df}{du}\overrightarrow{\nabla} u$

2. Soit $\overrightarrow{V} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k} \Longrightarrow u\overrightarrow{V} = uP\overrightarrow{i} + uQ\overrightarrow{j} + uR\overrightarrow{k}$

$\overrightarrow{\nabla} \cdot \left(u\overrightarrow{V}\right) = \dfrac{\partial(uP)}{\partial x} + \dfrac{\partial(uQ)}{\partial y} + \dfrac{\partial(uR)}{\partial z} = u\dfrac{\partial P}{\partial x} + \dfrac{\partial u}{\partial x}P + u\dfrac{\partial Q}{\partial y} + \dfrac{\partial u}{\partial y}Q + u\dfrac{\partial R}{\partial z} + \dfrac{\partial u}{\partial z}R$

$= u\left(\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + u\dfrac{\partial R}{\partial z}\right) + \dfrac{\partial u}{\partial x}P + \dfrac{\partial u}{\partial y}Q + \dfrac{\partial u}{\partial z}R$

$= u\overrightarrow{\nabla} \cdot \overrightarrow{V} + \overrightarrow{\nabla u} \cdot \overrightarrow{V}$

3. On utlise la relation $\overrightarrow{\nabla} f = \dfrac{df}{du}\overrightarrow{\nabla} u$ avec $f = r^n, u = r$

$\overrightarrow{\nabla} r^n = \dfrac{dr^n}{dr}\overrightarrow{\nabla} r = nr^{n-1}\overrightarrow{\nabla} r$

$\overrightarrow{\nabla} r = \dfrac{\partial r}{\partial x}\overrightarrow{i} + \dfrac{\partial r}{\partial y}\overrightarrow{j} + \dfrac{\partial r}{\partial z}\overrightarrow{k}$

$r=\sqrt{x^2+y^2+z^2}\Longrightarrow \frac{\partial r}{\partial x}=\frac{1}{2}(2x)\left(x^2+y^2+z^2\right)^{-1/2}=\frac{x}{r}$

de même $\frac{\partial r}{\partial y}=\frac{y}{r}$, et $\frac{\partial r}{\partial z}=\frac{z}{r}$

$\overrightarrow{\nabla}r=\frac{x}{r}\overrightarrow{i}+\frac{y}{r}\overrightarrow{j}+\frac{z}{r}\overrightarrow{k}=\frac{\overrightarrow{r}}{r}$

$$\overrightarrow{\nabla}r^n=nr^{n-1}\frac{\overrightarrow{r}}{r}=nr^{n-2}\overrightarrow{r}$$

4. (a) $n=1\Longrightarrow \overrightarrow{\nabla}r=r^{1-2}\overrightarrow{\nabla}r=\frac{\overrightarrow{r}}{r}$

(b) $n=-1\Longrightarrow \overrightarrow{\nabla}\left(\frac{1}{r}\right)=-1r^{-1-2}\overrightarrow{r}=-\frac{\overrightarrow{r}}{r^3}$

(c) $n=-2\Longrightarrow \overrightarrow{\nabla}\left(\frac{1}{r^2}\right)=-2r^{-2-2}\overrightarrow{r}=-2\frac{\overrightarrow{r}}{r^4}$

(d) $\Delta\left(\frac{1}{r}\right)=\overrightarrow{\nabla}\cdot\overrightarrow{\nabla}\left(\frac{1}{r}\right)=\overrightarrow{\nabla}\cdot\left(-\frac{\overrightarrow{r}}{r^3}\right)$

$=-\overrightarrow{\nabla}\left(\frac{1}{r^3}\right)\cdot\overrightarrow{r}-\frac{1}{r^3}\overrightarrow{\nabla}\cdot\overrightarrow{r}$

$=-\left(-3\frac{1}{r^5}\overrightarrow{r}\right)\cdot\overrightarrow{r}-\frac{3}{r^3}=3\frac{r^2}{r^5}-\frac{3}{r^3}=0$

5. D'après la formule de question 2) et la relation de question1)

$\overrightarrow{\nabla}\cdot\overrightarrow{H}=\overrightarrow{\nabla}\cdot(\overrightarrow{r}f(r))=f\overrightarrow{\nabla}\cdot\overrightarrow{r}+\overrightarrow{\nabla}f\cdot\overrightarrow{r}$

$=f\times(3)+\left(\frac{df}{dr}\overrightarrow{\nabla}r\right)\cdot\overrightarrow{r}$

$=3f+\frac{df}{dr}\frac{\overrightarrow{r}}{r}\cdot\overrightarrow{r}=3f+r\frac{df}{dr}$

$\overrightarrow{\nabla}\cdot\overrightarrow{H}=0\Longrightarrow 3f+r\frac{df}{dr}=0\Longrightarrow\frac{df}{f}=-3\frac{dr}{r}\Longrightarrow f(r)=\frac{K}{r^3}$

$f(1)=1\Longrightarrow K=1\Longrightarrow f(r)=\frac{1}{r^3}$

Exercice 14

Soit $f(r)$ une fonction deux fois dérivable par rapport à r.On désigne par $f'(r)$ et $f''(r)$ les dérivées première et seconde de $f(r)$.

1. Démontrer que $\overrightarrow{\text{grad}}\, f=f'(r)\frac{\overrightarrow{r}}{r}$
2. Calculer Δf et exprimer Δf en fonction de f' et f''
3. On pose $g=f'$,exprimer Δf en fonction de g et g', puis résoudre l'équation $\Delta f=0$.

Solution 14 :

1. $\overrightarrow{\nabla}f=\frac{\partial f}{\partial x}\overrightarrow{i}+\frac{\partial f}{\partial y}\overrightarrow{j}+\frac{\partial f}{\partial z}\overrightarrow{k}=\frac{df}{dr}\frac{\partial r}{\partial x}\overrightarrow{i}+\frac{df}{dr}\frac{\partial r}{\partial y}\overrightarrow{j}+\frac{df}{dr}\frac{\partial r}{\partial z}\overrightarrow{k}$

$=\frac{df}{dr}\left(\frac{\partial r}{\partial x}\overrightarrow{i}+\frac{\partial r}{\partial y}\overrightarrow{j}+\frac{\partial r}{\partial z}\overrightarrow{k}\right)=\frac{df}{dr}\overrightarrow{\nabla}r$

$$r=\sqrt{x^2+y^2+z^2}\Longrightarrow\frac{\partial r}{\partial x}=\frac{1}{2}(2x)\left(x^2+y^2+z^2\right)^{-1/2}=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}$$

de même $\frac{\partial r}{\partial y}=\frac{y}{r}$, et $\frac{\partial r}{\partial z}=\frac{z}{r}\Longrightarrow\overrightarrow{\nabla}r=\frac{x}{r}\overrightarrow{i}+\frac{y}{r}\overrightarrow{j}+\frac{z}{r}\overrightarrow{k}=\frac{\overrightarrow{r}}{r}$

d'où : $\overrightarrow{\nabla}f=\frac{df}{dr}\frac{\overrightarrow{r}}{r}$

2. $\Delta f=\overrightarrow{\nabla}\cdot\overrightarrow{\nabla}f=\overrightarrow{\nabla}\cdot\left(\frac{df}{dr}\frac{\overrightarrow{r}}{r}\right)=\overrightarrow{\nabla}\left(\frac{f'}{r}\right)\cdot\overrightarrow{r}+\frac{f'}{r}\overrightarrow{\nabla}\cdot\overrightarrow{r}$

$=\frac{1}{r}\overrightarrow{\nabla f'}\cdot\overrightarrow{r}+f'\overrightarrow{\nabla}\left(\frac{1}{r}\right)\cdot\overrightarrow{r}+\frac{f'}{r}\overrightarrow{\nabla}\cdot\overrightarrow{r}$

$\overrightarrow{\nabla f'}=\frac{df'}{dr}\frac{\overrightarrow{r}}{r}=f''\frac{\overrightarrow{r}}{r}\qquad\overrightarrow{\nabla}\left(\frac{1}{r}\right)=-\frac{\overrightarrow{r}}{r^3}\qquad\overrightarrow{\nabla}\cdot\overrightarrow{r}=3$

$\Delta f=\frac{1}{r}f''\frac{\overrightarrow{r}}{r}\cdot\overrightarrow{r}+f'\left(-\frac{\overrightarrow{r}}{r^3}\right)\cdot\overrightarrow{r}+3\frac{f'}{r}=f''+\frac{2}{r}f'$

3. $g=f'\Longrightarrow g'=f''\Longrightarrow\Delta f=g'+\frac{2}{r}g$

$\Delta f=0\Longleftrightarrow g'+\frac{2}{r}g=0$

$\Longrightarrow\frac{dg}{dr}=-2\frac{g}{r}\Longleftrightarrow\frac{dg}{g}=-2\frac{dr}{r}$

$\int\frac{dg}{g}=-2\int\frac{dr}{r}\Longrightarrow g=\frac{C}{r^2}$

$f(r)=C\int\frac{dr}{r^2}=-\frac{C}{r}+C_1$

Exercice 15

On considère les équations de Maxwell du champ électromagnétique

$$\overrightarrow{\nabla}\cdot\overrightarrow{H}=0\qquad\overrightarrow{\nabla}\times\overrightarrow{H}=\varepsilon\frac{\partial\overrightarrow{E}}{\partial t}$$
$$\overrightarrow{\nabla}\cdot\overrightarrow{E}=0\qquad\overrightarrow{\nabla}\times\overrightarrow{E}=-\mu\frac{\partial\overrightarrow{H}}{\partial t}$$

1. Calculer $\overrightarrow{\nabla}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)$
2. Montrer que $\overrightarrow{E}$ vérifie l'équation d'onde $\Delta\overrightarrow{E}-\frac{1}{v^2}\frac{\partial^2\overrightarrow{E}}{\partial t^2}=0$ avec $v=\frac{1}{\sqrt{\mu\varepsilon}}$.
3. Refaire le même calcul pour $\overrightarrow{H}$.
4. Vérifier que $\overrightarrow{E}=\overrightarrow{E_0}\exp\left(j\left(\omega t-\overrightarrow{k}\cdot\overrightarrow{r}\right)\right)$ est une solution de l'équation d'onde, avec ω et $\overrightarrow{E_0}$ sont constantes et $\overrightarrow{k}\,(k_x,k_y,k_z)$ tel que $k=\sqrt{k_x^2+k_y^2+k_z^2}=\frac{\omega}{v}$.

Solution 15 :

1. $\overrightarrow{\nabla}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)$
2. On a $\overrightarrow{\nabla}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)=\overrightarrow{\nabla}\left(\overrightarrow{\nabla}\cdot\overrightarrow{E}\right)-\Delta\overrightarrow{E}$

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\mu \frac{\partial \overrightarrow{H}}{\partial t} \Longrightarrow \overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{E}\right) = \overrightarrow{\nabla} \times \left(-\mu \frac{\partial \overrightarrow{H}}{\partial t}\right)$$

$$= -\mu \overrightarrow{\nabla} \times \frac{\partial \overrightarrow{H}}{\partial t} = -\mu \frac{\partial \overrightarrow{\nabla} \times \overrightarrow{H}}{\partial t} = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \overrightarrow{E}}{\partial t}\right) = -\mu\varepsilon \frac{\partial^2 \overrightarrow{E}}{\partial t^2} = -\frac{1}{v^2}\frac{\partial^2 \overrightarrow{E}}{\partial t^2}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0 \Longrightarrow \overrightarrow{\nabla}\left(\overrightarrow{\nabla} \cdot \overrightarrow{E}\right) = \overrightarrow{0}$$

$$\Longrightarrow \overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{E}\right) = -\Delta \overrightarrow{E} = -\frac{1}{v^2}\frac{\partial^2 \overrightarrow{E}}{\partial t^2}$$

d'où :

$$\Delta \overrightarrow{E} - \frac{1}{v^2}\frac{\partial^2 \overrightarrow{E}}{\partial t^2} = 0$$

3. $\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{H}\right) = \overrightarrow{\nabla}\left(\overrightarrow{\nabla} \cdot \overrightarrow{H}\right) - \Delta \overrightarrow{H} = -\Delta \overrightarrow{H}$

$$\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{H}\right) = \overrightarrow{\nabla} \times \left(\varepsilon \frac{\partial \overrightarrow{E}}{\partial t}\right) = \varepsilon \frac{\partial \overrightarrow{\nabla} \times \overrightarrow{E}}{\partial t}$$

$$= \varepsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \overrightarrow{H}}{\partial t}\right) = -\varepsilon\mu \frac{\partial^2 \overrightarrow{H}}{\partial t^2} = -\frac{1}{v^2}\frac{\partial^2 \overrightarrow{H}}{\partial t^2} = -\Delta \overrightarrow{H}$$

$$\Delta \overrightarrow{H} - \frac{1}{v^2}\frac{\partial^2 \overrightarrow{H}}{\partial t^2} = 0$$

4. $\overrightarrow{E} = \overrightarrow{E_0} \exp\left(j\left(\omega t - \overrightarrow{k} \cdot \overrightarrow{r}\right)\right) = \overrightarrow{E_0} \exp\left(j\left(\omega t - xk_x - yk_y - zk_z\right)\right)$

$$\frac{\partial \overrightarrow{E}}{\partial x} = -jk_x \overrightarrow{E} \Longrightarrow \frac{\partial^2 \overrightarrow{E}}{\partial x^2} = -k_x^2 \overrightarrow{E}$$

de même : $\frac{\partial^2 \overrightarrow{E}}{\partial y^2} = -k_y^2 \overrightarrow{E}$ et $\frac{\partial^2 \overrightarrow{E}}{\partial z^2} = -k_z^2 \overrightarrow{E}$ et $\frac{\partial^2 \overrightarrow{E}}{\partial t^2} = -\omega^2 \overrightarrow{E}$

$$\Delta \overrightarrow{E} = \frac{\partial^2 \overrightarrow{E}}{\partial x^2} + \frac{\partial^2 \overrightarrow{E}}{\partial y^2} + \frac{\partial^2 \overrightarrow{E}}{\partial z^2} = -\left(k_x^2 + k_y^2 + k_z^2\right)\overrightarrow{E} = -k^2 \overrightarrow{E} = -\frac{\omega^2}{v^2}\overrightarrow{E} = \frac{1}{v^2}\frac{\partial^2 \overrightarrow{E}}{\partial t^2}$$

Exercice 16

Les équations de Maxwell du champ électromagnétique dans le vide sont

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0 \qquad \overrightarrow{\nabla} \times \overrightarrow{B} = \frac{1}{c}\frac{\partial \overrightarrow{E}}{\partial t}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0 \qquad \overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{1}{c}\frac{\partial \overrightarrow{B}}{\partial t}$$

Soient $\overrightarrow{A}$ un champ de vecteurs et φ un champ scalaire tels que :

$$\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A} \quad et \quad \overrightarrow{E} = -\overrightarrow{\nabla}\varphi - \frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t}$$

1. Sachant que : $\overrightarrow{\nabla} \cdot \overrightarrow{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0$, démontrer que : $\overrightarrow{A}$ et φ vérifient l'équation

$$\Delta \alpha - \frac{1}{c^2}\frac{\partial^2 \alpha}{\partial t^2} = 0$$

2. En un point $M(x,y,z)$ on pose $\overrightarrow{E} = E_x\overrightarrow{e_x} + E_y\overrightarrow{e_y} + E_z\overrightarrow{e_z}$ tel que :

$$E_x = \frac{x}{x^2+y^2}\cos(\omega t - kz)$$
$$E_y = \frac{y}{x^2+y^2}\cos(\omega t - kz)$$
$$E_z = 0$$

où k et ω sont deux consatntes positives. et $c = \frac{\omega}{k}$

Le champ magnétique en un point $M(x,y,z)$ est défini par :

$$\overrightarrow{B} = \frac{k}{\omega}\left(E_y\overrightarrow{e_x} - E_x\overrightarrow{e_y}\right)$$

(a) Montrer que $\overrightarrow{E}$ et $\overrightarrow{B}$ vérifient les équations de Maxwell

(b) En utilisant la relation

$$\overrightarrow{\nabla}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right) = \overrightarrow{\nabla}\left(\overrightarrow{\nabla}\cdot\overrightarrow{E}\right) - \Delta\overrightarrow{E}$$

Montrer que $\overrightarrow{E}$ vérifie l'équation de propagation d'onde :

$$\Delta\overrightarrow{E} + k^2\overrightarrow{E} = 0$$

Solution 16 :

1. $\overrightarrow{\nabla}\times\overrightarrow{B} = \frac{1}{c}\frac{\partial\overrightarrow{E}}{\partial t}$ et $\overrightarrow{B} = \overrightarrow{\nabla}\times\overrightarrow{A}$ et $\overrightarrow{E} = -\overrightarrow{\nabla}\varphi - \frac{1}{c}\frac{\partial\overrightarrow{A}}{\partial t} \Longrightarrow$

$$\overrightarrow{\nabla}\times\left(\overrightarrow{\nabla}\times\overrightarrow{A}\right) = \frac{1}{c}\frac{\partial\overrightarrow{E}}{\partial t}$$

$\overrightarrow{\nabla}\times\left(\overrightarrow{\nabla}\times\overrightarrow{A}\right) = \overrightarrow{\nabla}\left(\overrightarrow{\nabla}\cdot\overrightarrow{A}\right) - \Delta\overrightarrow{A}$

$\frac{1}{c}\frac{\partial\overrightarrow{E}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\overrightarrow{\nabla}\varphi - \frac{1}{c}\frac{\partial\overrightarrow{A}}{\partial t}\right)$

$\Longrightarrow$

$$\overrightarrow{\nabla}\left(\overrightarrow{\nabla}\cdot\overrightarrow{A}\right) - \Delta\overrightarrow{A} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\overrightarrow{\nabla}\varphi - \frac{1}{c}\frac{\partial\overrightarrow{A}}{\partial t}\right)$$

$\overrightarrow{\nabla}\cdot\overrightarrow{A} = -\frac{1}{c}\frac{\partial\varphi}{\partial t}$ et $\frac{\partial}{\partial t}\left(\overrightarrow{\nabla}\varphi\right) = \overrightarrow{\nabla}\left(\frac{\partial\varphi}{\partial t}\right)$

L'équation (2) devient

$\overrightarrow{\nabla}\left(-\frac{1}{c}\frac{\partial\varphi}{\partial t}\right) - \Delta\overrightarrow{A} = -\frac{1}{c}\overrightarrow{\nabla}\left(\frac{\partial\varphi}{\partial t}\right) - \frac{1}{c^2}\frac{\partial\overrightarrow{A}}{\partial t}$ en simplifiant par $-\frac{1}{c}\overrightarrow{\nabla}\left(\frac{\partial\varphi}{\partial t}\right)$ on trouve

$$\Delta\overrightarrow{A} - \frac{1}{c^2}\frac{\partial^2\overrightarrow{A}}{\partial t^2} = 0 \qquad (2)$$

$\overrightarrow{\nabla}\cdot\overrightarrow{E} = 0 \Longrightarrow \overrightarrow{\nabla}\cdot\left(-\overrightarrow{\nabla}\varphi - \frac{1}{c}\frac{\partial\overrightarrow{A}}{\partial t}\right) = 0$

$\Longrightarrow -\Delta\varphi - \frac{1}{c}\overrightarrow{\nabla}\cdot\left(\frac{\partial\overrightarrow{A}}{\partial t}\right) = 0$

$$\Longrightarrow \Delta\varphi + \frac{1}{c}\frac{\partial}{\partial t}\left(\overrightarrow{\nabla}\cdot\overrightarrow{A}\right) = 0 \Longrightarrow \Delta\varphi + \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\varphi}{\partial t}\right) = 0 \Longrightarrow$$

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0$$

2. $\overrightarrow{E} = E_x\overrightarrow{e_x} + E_y\overrightarrow{e_y} + E_z\overrightarrow{e_z}$

(a)

$$\left.\begin{aligned}\frac{\partial E_x}{\partial x} &= \frac{\partial}{\partial x}\left(\tfrac{x}{x^2+y^2}\cos(\omega t - kz)\right) = \tfrac{-x^2+y^2}{(x^2+y^2)^2}\cos(\omega t - kz)\\ \frac{\partial E_y}{\partial y} &= \frac{\partial}{\partial y}\left(\tfrac{y}{x^2+y^2}\cos(\omega t - kz)\right) = \tfrac{x^2-y^2}{(x^2+y^2)^2}\cos(\omega t - kz)\\ \frac{\partial E_z}{\partial z} &= 0\end{aligned}\right\}$$

$$\Longrightarrow \overrightarrow{\nabla}\cdot\overrightarrow{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

$$\left.\begin{aligned}\frac{\partial B_x}{\partial x} &= \frac{k}{\omega}\frac{\partial}{\partial x}\left(-\tfrac{y}{x^2+y^2}\cos(\omega t - kz)\right) = \frac{k}{\omega}\tfrac{2xy}{(x^2+y^2)^2}\cos(\omega t - kz)\\ \frac{\partial B_y}{\partial y} &= \frac{k}{\omega}\frac{\partial}{\partial y}\left(\tfrac{x}{x^2+y^2}\cos(\omega t - kz)\right) = -\frac{k}{\omega}\tfrac{2xy}{(x^2+y^2)^2}\cos(\omega t - kz)\\ \frac{\partial B_z}{\partial z} &= 0\end{aligned}\right\}$$

$$\Longrightarrow \overrightarrow{\nabla}\cdot\overrightarrow{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$\overrightarrow{\nabla}\times\overrightarrow{E} = -\frac{ky}{x^2+y^2}\sin(\omega t - kz)\,\overrightarrow{e_x} + \frac{kx}{x^2+y^2}\sin(\omega t - kz)\,\overrightarrow{e_y}$$

$$\frac{\partial\overrightarrow{B}}{\partial t} = \frac{k}{\omega}\frac{\partial}{\partial t}\left(\left(-E_y\overrightarrow{e_x} + E_x\overrightarrow{e_y}\right)\right) = \frac{k}{\omega}\left(\left(-\frac{\partial E_y}{\partial t}\overrightarrow{e_x} + \frac{\partial E_x}{\partial t}\overrightarrow{e_y}\right)\right)$$

$$= \frac{k}{\omega}\left(\frac{\omega y}{x^2+y^2}\sin(\omega t - kz)\,\overrightarrow{e_x} - \frac{\omega x}{x^2+y^2}\sin(\omega t - kz)\,\overrightarrow{e_y}\right)$$

$$= -\overrightarrow{\nabla}\times\overrightarrow{E}$$

$$\overrightarrow{\nabla}\times\overrightarrow{B} = -\frac{k^2}{\omega}\left(\frac{x}{x^2+y^2}\sin(\omega t - kz)\,\overrightarrow{e_x} + \frac{y}{x^2+y^2}\sin(\omega t - kz)\,\overrightarrow{e_y}\right) = \frac{k^2}{\omega^2}\frac{\partial\overrightarrow{E}}{\partial t}$$

(b) $\overrightarrow{\nabla}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right) = \overrightarrow{\nabla}\left(\overrightarrow{\nabla}\cdot\overrightarrow{E}\right) - \Delta\overrightarrow{E}$

$$\Longleftrightarrow \overrightarrow{\nabla}\times\left(-\frac{\partial\overrightarrow{B}}{\partial t}\right) = \overrightarrow{\nabla}(0) - \Delta\overrightarrow{E}$$

$$\Longleftrightarrow -\frac{\partial}{\partial t}\left(\overrightarrow{\nabla}\times\overrightarrow{B}\right) = -\Delta\overrightarrow{E}$$

$$\Longleftrightarrow -\frac{\partial}{\partial t}\left(\frac{k^2}{\omega^2}\frac{\partial\overrightarrow{E}}{\partial t}\right) = -\Delta\overrightarrow{E}$$

$$\Longleftrightarrow -\frac{k^2}{\omega^2}\frac{\partial^2\overrightarrow{E}}{\partial t^2} = -\Delta\overrightarrow{E}$$

$$\Longleftrightarrow -\frac{k^2}{\omega^2}\left(-\omega^2\overrightarrow{E}\right) = -\Delta\overrightarrow{E}$$

Finalement

$$\Delta\overrightarrow{E} + k^2\overrightarrow{E} = 0$$

Exercice 17

En utilisant les dérivées en chaine des fonctions composées, exprimer les opérateurs $\overrightarrow{\nabla}$ et Δ en coordonnées cylindriques.

Solution 17 On a

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \\ z = z \end{cases}$$

Soit $g = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$

$r = \sqrt{x^2+y^2} \Longrightarrow \frac{\partial r}{\partial x} = \frac{1}{2}(2x)\left(x^2+y^2\right)^{-1/2} = \frac{x}{r} = \cos\theta$

$\theta = \arctan\frac{y}{x} \Longrightarrow \frac{\partial \theta}{\partial x} = \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$

$\frac{\partial z}{\partial x} = 0$

$$g = \frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

Posons $h = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$

$r = \sqrt{x^2+y^2} \Longrightarrow \frac{\partial r}{\partial y} = \frac{1}{2}(2y)\left(x^2+y^2\right)^{-1/2} = \frac{y}{r} = \sin\theta$

$\theta = \arctan\frac{y}{x} \Longrightarrow \frac{\partial \theta}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$

$\frac{\partial z}{\partial y} = 0$

$$h = \frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\begin{aligned}
\overrightarrow{\nabla} f &= \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j} + \frac{\partial f}{\partial z}\overrightarrow{k} = \frac{\partial f}{\partial r}\overrightarrow{e_r} + \frac{\partial f}{\partial \theta}\overrightarrow{e_\theta} + \frac{\partial f}{\partial z}\overrightarrow{k} \\
&= \left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right)\overrightarrow{i} + \left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right)\overrightarrow{j} + \frac{\partial f}{\partial z}\overrightarrow{k} \\
&= \frac{\partial f}{\partial r}\left(\cos\theta\,\overrightarrow{i} + \sin\theta\,\overrightarrow{j}\right) + \frac{1}{r}\frac{\partial f}{\partial \theta}\left(-\sin\theta\,\overrightarrow{i} + \cos\theta\,\overrightarrow{j}\right) + \frac{\partial f}{\partial z}\overrightarrow{k}
\end{aligned}$$

Soit en introduisant $\left(\overrightarrow{e_r}, \overrightarrow{e_\theta}, \overrightarrow{k}\right)$

$$\overrightarrow{\nabla} f = \frac{\partial f}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\overrightarrow{e_\theta} + \frac{\partial f}{\partial z}\overrightarrow{k}$$

La divergence

Soit $\overrightarrow{H}$ un champ vectoriel défini, en coordonnées cylindriques, par

$$\overrightarrow{H} = R\overrightarrow{e_r} + T\overrightarrow{e_\theta} + Z\overrightarrow{k}$$

où R, T et Z sont des fonctions scalaires de (r, θ, z).

L'opérateur $\overrightarrow{\nabla}$ s'écrit en coordonnées cylindriques :

$$\overrightarrow{\nabla} = \frac{\partial}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\overrightarrow{e_\theta} + \frac{\partial}{\partial z}\overrightarrow{k}$$

alors

$$\text{div}\,\overrightarrow{H} = \overrightarrow{\nabla}\cdot\overrightarrow{H} = \left(\frac{\partial}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\overrightarrow{e_\theta} + \frac{\partial}{\partial z}\overrightarrow{k}\right)\cdot\overrightarrow{H}$$

Notons que $\overrightarrow{e_r}$ et $\overrightarrow{e_\theta}$ ne sont pas des vecteurs constantes, ils sont des fonctions de θ :

$$\overrightarrow{e_r} = \cos\theta\,\overrightarrow{i} + \sin\theta\,\overrightarrow{j}$$
$$\overrightarrow{e_\theta} = -\sin\theta\,\overrightarrow{i} + \cos\theta\,\overrightarrow{j}$$

on a donc :

$$\frac{\partial\overrightarrow{e_r}}{\partial\theta} = \overrightarrow{e_\theta}\ et\ \frac{\partial\overrightarrow{e_\theta}}{\partial\theta} = -\overrightarrow{e_r}$$

$$\begin{aligned}
\overrightarrow{\nabla}\cdot\overrightarrow{H} &= \frac{\partial\overrightarrow{H}}{\partial r}\cdot\overrightarrow{e_r} + \frac{1}{r}\frac{\partial\overrightarrow{H}}{\partial\theta}\cdot\overrightarrow{e_\theta} + \frac{\partial\overrightarrow{H}}{\partial z}\cdot\overrightarrow{k}\\
&= \frac{\partial R}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(R\overrightarrow{e_r} + T\overrightarrow{e_\theta}\right)\cdot\overrightarrow{e_\theta} + \frac{\partial Z}{\partial z}\\
&= \frac{\partial R}{\partial r} + \frac{1}{r}\left(\frac{\partial R}{\partial\theta}\overrightarrow{e_r} + R\frac{\partial\overrightarrow{e_r}}{\partial\theta} + \frac{\partial T}{\partial\theta}\overrightarrow{e_\theta} + T\frac{\partial\overrightarrow{e_\theta}}{\partial\theta}\right)\cdot\overrightarrow{e_\theta} + \frac{\partial Z}{\partial z}\\
&= \frac{\partial R}{\partial r} + \frac{1}{r}\left(\frac{\partial R}{\partial\theta}\overrightarrow{e_r} + R\overrightarrow{e_\theta} + \frac{\partial T}{\partial\theta}\overrightarrow{e_\theta} - T\overrightarrow{e_r}\right)\cdot\overrightarrow{e_\theta} + \frac{\partial Z}{\partial z}\\
&= \frac{\partial R}{\partial r} + \frac{1}{r}\left(R + \frac{\partial T}{\partial\theta}\right) + \frac{\partial Z}{\partial z}
\end{aligned}$$

Comme $\frac{1}{r}\frac{\partial(rR)}{\partial r} = \frac{\partial R}{\partial r} + \frac{R}{r}$ on écrit :

$$\overrightarrow{\nabla}\cdot\overrightarrow{H} = \frac{1}{r}\frac{\partial(rR)}{\partial r} + \frac{1}{r}\frac{\partial T}{\partial\theta} + \frac{\partial Z}{\partial z} \tag{3}$$

Le laplacien
Par définition :

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

On a :
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}$$

$$\frac{\partial g}{\partial r} = \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) = \cos\theta\frac{\partial^2 f}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} + \frac{\sin\theta}{r^2}\frac{\partial f}{\partial\theta}$$

$$\frac{\partial g}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) = -\sin\theta\frac{\partial f}{\partial r} + \cos\theta\frac{\partial^2 f}{\partial r\partial\theta} - \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial\theta^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \cos\theta\left(\cos\theta\frac{\partial^2 f}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} + \frac{\sin\theta}{r^2}\frac{\partial f}{\partial\theta}\right) - \frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial f}{\partial r} + \cos\theta\frac{\partial^2 f}{\partial r\partial\theta} - \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial\theta^2}\right)$$

$$= \cos^2\theta\frac{\partial^2 f}{\partial r^2} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

Soit :

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r}$$

$\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial y}{\partial y} = \dfrac{\partial h}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial h}{\partial \theta}\dfrac{\partial \theta}{\partial y} + \dfrac{\partial h}{\partial z}\dfrac{\partial z}{\partial y}$

$$\frac{\partial^2 f}{\partial y^2} = \sin\theta\frac{\partial h}{\partial r} + \frac{\cos\theta}{r}\frac{\partial h}{\partial \theta}$$

$\dfrac{\partial h}{\partial r} = \dfrac{\partial}{\partial r}\left(\sin\theta\dfrac{\partial f}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial f}{\partial \theta}\right) = \sin\theta\dfrac{\partial^2 f}{\partial r^2} + \dfrac{\cos\theta}{r}\dfrac{\partial^2 f}{\partial\theta\partial r} - \dfrac{\cos\theta}{r^2}\dfrac{\partial f}{\partial \theta}$

$\dfrac{\partial h}{\partial \theta} = \dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial f}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial f}{\partial \theta}\right) = \cos\theta\dfrac{\partial f}{\partial r} + \sin\theta\dfrac{\partial^2 f}{\partial r\partial\theta} + \dfrac{\cos\theta}{r}\dfrac{\partial^2 f}{\partial \theta^2} - \dfrac{\sin\theta}{r}\dfrac{\partial f}{\partial \theta}$

$\dfrac{\partial^2 f}{\partial y^2} = \sin\theta\left(\sin\theta\dfrac{\partial^2 f}{\partial r^2} + \dfrac{\cos\theta}{r}\dfrac{\partial^2 f}{\partial\theta\partial r} - \dfrac{\cos\theta}{r^2}\dfrac{\partial f}{\partial \theta}\right) + \dfrac{\cos\theta}{r}\left(\cos\theta\dfrac{\partial f}{\partial r} + \sin\theta\dfrac{\partial^2 f}{\partial r\partial\theta} + \dfrac{\cos\theta}{r}\dfrac{\partial^2 f}{\partial \theta^2} - \dfrac{\sin\theta}{r}\dfrac{\partial f}{\partial \theta}\right)$

$= \quad \sin^2\theta\dfrac{\partial^2 f}{\partial r^2} + \dfrac{\sin\theta\cos\theta}{r}\dfrac{\partial^2 f}{\partial\theta\partial r} - \dfrac{\sin\theta\cos\theta}{r^2}\dfrac{\partial f}{\partial \theta} + \dfrac{\cos^2\theta}{r}\dfrac{\partial f}{\partial r} + \dfrac{\sin\theta\cos\theta}{r}\dfrac{\partial^2 f}{\partial r\partial\theta} + \dfrac{\cos^2\theta}{r^2}\dfrac{\partial^2 f}{\partial \theta^2} - \dfrac{\cos\theta\sin\theta}{r^2}\dfrac{\partial f}{\partial \theta}$

Soit :

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r}$$

Substituons dans Δf :

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
$$= \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r}$$
$$+\sin^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2}$$

Finalement :

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

L'expression de Δf en coordonnées cylindriques, peut être obtenue aussi de la définition : $\Delta = \vec{\nabla}\cdot\left(\vec{\nabla} f\right)$

$\Delta f = \left(\dfrac{\partial}{\partial r}\vec{e_r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\vec{e_\theta} + \dfrac{\partial}{\partial z}\vec{k}\right)\cdot\left(\vec{\nabla} f\right) = \dfrac{\partial}{\partial r}\left(\vec{\nabla} f\right)\cdot\vec{e_r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\left(\vec{\nabla} f\right)\cdot\vec{e_\theta} + \dfrac{\partial}{\partial z}\left(\vec{\nabla} f\right)\cdot\vec{k}$

$$\frac{\partial}{\partial r}\left(\vec{\nabla} f\right)\cdot\vec{e_r} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{\partial f}{\partial z}\vec{k}\right)\cdot\vec{e_r}$$
$$= \left(\frac{\partial^2 f}{\partial r^2}\vec{e_r} + \frac{1}{r}\frac{\partial^2 f}{\partial r\partial\theta}\vec{e_\theta} - \frac{1}{r^2}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{\partial^2 f}{\partial r\partial z}\vec{k}\right)\cdot\vec{e_r} = \frac{\partial^2 f}{\partial r^2}$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(\vec{\nabla} f\right)\cdot\vec{e_\theta} = \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{\partial f}{\partial z}\vec{k}\right)\cdot\vec{e_\theta}$$
$$= \frac{1}{r}\left(\frac{\partial^2 f}{\partial\theta\partial r}\vec{e_r} + \frac{\partial f}{\partial r}\frac{\partial\vec{e_r}}{\partial \theta} + \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2}\vec{e_\theta} + \frac{1}{r}\frac{\partial f}{\partial \theta}\frac{\partial\vec{e_\theta}}{\partial \theta} + \frac{\partial^2 f}{\partial\theta\partial z}\vec{k}\right)\cdot\vec{e_\theta}$$
$$= \frac{1}{r}\left(\frac{\partial^2 f}{\partial\theta\partial r}\vec{e_r} + \frac{\partial f}{\partial r}\vec{e_\theta} + \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2}\vec{e_\theta} - \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_r} + \frac{\partial^2 f}{\partial\theta\partial z}\vec{k}\right)\cdot\vec{e_\theta} = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial}{\partial z}\left(\vec{\nabla} f\right)\cdot\vec{k} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{\partial f}{\partial z}\vec{k}\right)\cdot\vec{k} = \frac{\partial^2 f}{\partial z^2}$$

d'où

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$