Fonctions de plusieures variables

Exercices corrigés

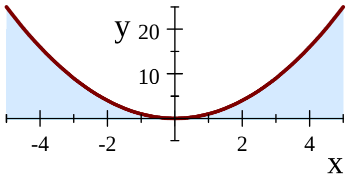
Exercice 1

Dans chaque cas, déterminez et représentez le domaine de définition des fonctions données

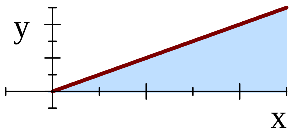
1. $f(x,y) = \dfrac{\sqrt{x^2-y}}{\sqrt{y}}$
2. $f(x,y) = \dfrac{\ln y}{\sqrt{x-y}}$
3. $f(x,y) = \ln(x+y)$
4. $f(x,y,z) = \ln(x+y+z)$
5. $f(x,y,z) = \dfrac{\ln(x^2+1)}{yz}$
6. $f(x,y,z) = \sqrt{1-x^2-y^2-z^2}$

Solution 1 :

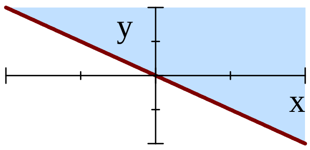
1. $f(x,y)$ est définie si $x^2 - y \geq 0$ et $y > 0 \Longrightarrow D = \{(x,y) \in \mathbb{R}^2 | \, y \leq x^2 \text{ et } y > 0\}$



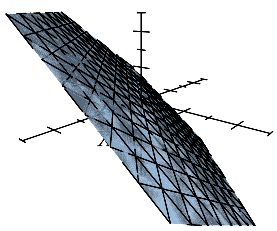
2. $D = \{(x,y) \in \mathbb{R}^2 | \, x > y \text{ et } y > 0\}$



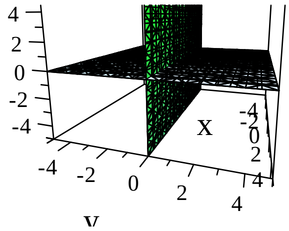
3. $D = \{(x,y) \in \mathbb{R}^2 | \, x > -y\}$



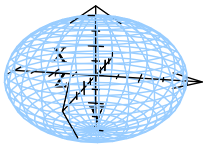
4. $D = \{(x,y,z) \in \mathbb{R}^3 | \, x+y+z > 0\}$
   la partie de l'espace au dessus du plan $x+y+z=0$



5. $D = \{ (x,y,z) \in \mathbb{R}^3 | \, y \neq 0, z \neq 0 \}$, la fonction $f(x,y,z) = \dfrac{\ln(x^2+1)}{yz}$ est définie dans tout l'espace sauf les plans (xOz) et (yOz)



6. $D = \{ (x,y,z) \in \mathbb{R}^3 | \, x^2+y^2+z^2 \leq 1 \}$
Le domaine de définition est à l'intérieure de la sphère centrée en O et de rayon 1



Exercice 2

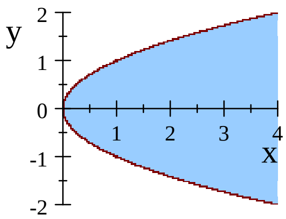
Soit la fonction $f : \mathbb{R}^2 \to \mathbb{R}$ définie par :

$$f(x,y) = \ln(x - y^2)$$

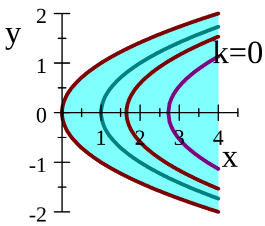
1. Déterminer et représenter le domaine de définition de la fonction f
2. Déterminer et représenter ses courbes de niveau

Solution 2 : $f(x,y) = \ln(x - y^2)$

1. La fonction est définie si $x - y^2 > 0$, donc $D = \{ (x,y) \in \mathbb{R}^2 | \, x > y^2 \}$
2. Les courbes de niveaux sont telles que $f(x,y) = \ln(x - y^2) = k = cte$
soit $x - y^2 = e^k$ ou bien $x = y^2 + e^k$



D : $x > y^2$



courbes de niveaux :
$k = 1, \dfrac{1}{2}, 0$ et $-\infty$

Exercice 3

Calculer la limite, si elle existe, au point $M(x_0, y_0)$ de la fonction $f(x,y)$ dans les cas suivants :

1) $\dfrac{xy}{x^2+y^2}$	$M_0(0,0)$	;	2) $\dfrac{xy^2}{x^2+y^4}$	$M_0(0,0)$
3) $\dfrac{x^2y}{4x^2-y}$	$M_0(1,3)$	;	4) $\dfrac{xy^2}{x^2+y^2}$	$M_0(0,0)$
5) $\dfrac{x^2y}{x^2+y^2}$	$M_0(0,0)$	;	6) $\dfrac{2x^2\sin y}{2x^2+y^2}$	$M_0(0,0)$
7) $\dfrac{\sin(x^2+y^2)}{x^2+y^2}$	$M_0(0,0)$	;	8) $\dfrac{\cos xy}{1+y^2}$	$M_0(\pi,1)$
9) $\dfrac{x^3+4x^2+2y^2}{2x^2+y^2}$	$M_0(0,0)$	;	10) $\dfrac{xy-2x-y+2}{x^2+y^2-2x-4y+5}$	$M_0(1,2)$

Solution 3 :

1. $f(x,y) = \dfrac{xy}{x^2+y^2}, \qquad M_0(0,0)$

   Considérons 3 trajets :

   (I) sur $x=0$ on a $f(0,y) = \dfrac{0}{0+y^2} = 0$

   donc $\lim\limits_{M\to M_0} f(x,y)\Big|_I = \lim\limits_{y\to 0} f(0,y) = 0$

   (II) Suivant $y=0$ on a $f(x,0) = \dfrac{0}{x^2+0} = 0$

   $\Longrightarrow \lim\limits_{M\to M_0} f(x,y)\Big|_{II} = \lim\limits_{x\to 0} f(x,0) = 0$

   (III) Suivant le trajet $x=y$: $f(x,x) = \dfrac{x^2}{x^2+x^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$

   $\Longrightarrow \lim\limits_{M\to M_0} f(x,y)\Big|_{III} = \lim\limits_{(x,y)\to(0,0)} \dfrac{1}{2} = \dfrac{1}{2}$

   Finalement la limite n'est pas la même alors pas de limite au point $M_0(0,0)$

2. $f(x,y) = \dfrac{xy^2}{x^2+y^4}, \qquad M_0(0,0)$

   (I) $x=0 \Longrightarrow f(0,y) = \dfrac{0}{0+y^4} = 0$

   $\Longrightarrow \lim\limits_{M\to M_0} f(x,y)\Big|_I = \lim\limits_{y\to 0} f(0,y) = 0$

   (II) $y=0 \Longrightarrow f(x,0) = \dfrac{0}{x^2+0} = 0$

   $\Longrightarrow \lim\limits_{M\to M_0} f(x,y)\Big|_{II} = \lim\limits_{x\to 0} f(x,0) = 0$

   (III) $x=y \Longrightarrow f(x,x) = \dfrac{x^3}{x^2+x^4} = \dfrac{x}{1+x^2} \underset{x\to 0}{\longrightarrow} 0$

   $\Longrightarrow \lim\limits_{M\to M_0} f(x,y)\Big|_{III} = \lim\limits_{x\longrightarrow 0} f(x,x) = 0$

   (IV) $x=y^2 \Longrightarrow f(y^2,y) = \dfrac{y^4}{y^4+y^4} = \dfrac{1}{2}$

   $\Longrightarrow \lim\limits_{M\to M_0} f(x,y)\Big|_{IV} = \lim\limits_{(x,y)\to(0,0)} \dfrac{1}{2} = \dfrac{1}{2}$

   pas de limite au point $(0,0)$

3. $f(x,y) = \dfrac{x^2 y}{4x^2 - y} \qquad M_0(1,3)$

   La fonction est définie au point $M_0(1,3)$ alors la limite est :

   $f(1,3) = \dfrac{1 \times 3}{4 \times 1 - 3} = 3$

4. $f(x,y) = \dfrac{xy^2}{x^2+y^2}, \qquad M_0(0,0)$

   (I) pour $x = 0$ on a $f(0,y) = 0 \Longrightarrow \lim\limits_{M \to M_0} f(x,y)\Big|_I = 0$

   (II) pour $y = 0$ on a $f(x,0) = 0 \Longrightarrow \lim\limits_{M \to M_0} f(x,y)\Big|_{II} = 0$

   (III) Si $x = y$: $f(x,x) = \dfrac{x^3}{2x^2} = \dfrac{x}{2} \Longrightarrow \lim\limits_{M \to M_0} f(x,y)\Big|_{III} = \lim\limits_{x \to 0} \dfrac{x}{2} = 0$

   (IV) Pour $y = x^2$: $f(x,x^2) = \dfrac{x^5}{x^2+x^4} = \dfrac{x^3}{1+x^2}$

   $\Longrightarrow \lim\limits_{M \to M_0} f(x,y)\Big|_{IV} = \lim\limits_{x \to 0} \dfrac{x^3}{1+x^2} = 0$

   Démontrons que $\lim\limits_{M \to M_0} f(x,y) = 0$

   Si L est la limite de $f(x,y)$ au point M_0 alors que $|f(x,y) - L| \leq g(x,y)$ où $g(x,y)$ est une fonction qui tend vers *zéro* quand $M \to M_0$.

   on a $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ et comme $x^2 + y^2 \geq y^2 \Longrightarrow \dfrac{1}{x^2+y^2} \leq \dfrac{1}{y^2}$

   par suite $\left|\dfrac{xy^2}{x^2+y^2} - 0\right| \leq \left|\dfrac{xy^2}{y^2}\right| = |x|$

   alors $g(x,y) = x \to 0$ si $M(x,y) \to M_0(0,0)$

   Finalement $\lim\limits_{M \to M_0} f(x,y) = 0$

5. $f(x,y) = \dfrac{x^2 y}{x^2+y^2} \qquad M_0(0,0)$

   par analogie avec 4) On a $f(0,y) = f(x,0) = 0$ et pour $x = y$ on a

   $f(x,x) = \dfrac{x}{2} \underset{x \to 0}{\longrightarrow} 0$ et $f(y^2,y) = \dfrac{y^5}{y^4+y^2} = \dfrac{y^3}{y^2+1} \underset{x \to 0}{\longrightarrow} 0$

   Montrons qu'il existe $g(x,y)$ telle que $|f(x,y) - L| \leq g(x,y)$ et $g(x,y) \underset{x \to 0}{\longrightarrow} 0$

   $x^2 + y^2 \geq x^2 \Longrightarrow \dfrac{1}{x^2+y^2} \leq \dfrac{1}{x^2}$

   $\Longrightarrow |f(x,y) - 0| = \left|\dfrac{x^2 y}{x^2+y^2}\right| \leq |y| = g(x,y) \underset{x \to 0}{\longrightarrow} 0$

6. $f(x,y) = \dfrac{2x^2 \sin y}{2x^2+y^2} \qquad M_0(0,0)$

   $f(0,y) = f(x,0) = 0$ Montrons que la limite est $= 0$

   $2x^2 \leq 2x^2 + y^2 \Longrightarrow 2x^2 |\sin y| \leq (2x^2+y^2)|\sin y|$

   $\left|\dfrac{2x^2 \sin y}{2x^2+y^2} - 0\right| = \dfrac{2x^2 |\sin y|}{2x^2+y^2} \leq |\sin y| = g(x,y) \underset{x \to 0}{\longrightarrow} 0$

   $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2} \qquad M_0(0,0)$

   On sait que $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ il suffit de poser $\theta = x^2 + y^2$

   $\theta \to 0$ si $(x,y) \to (0,0)$

alors on écrit $f(x,y) = \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \dfrac{\sin\theta}{\theta} \underset{x\to 0}{\longrightarrow} 0$

7. $f(x,y) = \dfrac{x^3+4x^2+2y^2}{2x^2+y^2} \qquad M_0(0,0)$

(I) $f(x,0) = \dfrac{x^3+4x^2}{2x^2} = \dfrac{x+4}{2} \Longrightarrow \lim\limits_{M\to M_0} f(x,y)\Big|_I = \lim\limits_{x\to 0}\left(\dfrac{x+4}{2}\right) = 2$

(II) $f(0,y) = \dfrac{2y^2}{y^2} = 2 \Longrightarrow \lim\limits_{M\to M_0} f(x,y)\Big|_{II} = \lim\limits_{x\to 0}(2) = 2$

$\left|\dfrac{x^3+4x^2+2y^2}{2x^2+y^2} - 2\right| = \left|\dfrac{x^3+4x^2+2y^2-4x^2-2y^2}{2x^2+y^2}\right| = \left|\dfrac{x^3}{2x^2+y^2}\right|$

$x^2 \le 2x^2+y^2 \Longrightarrow \dfrac{x^2}{2x^2+y^2} \le 1$

$\left|\dfrac{x^3}{2x^2+y^2}\right| = \dfrac{|x|\,x^2}{2x^2+y^2} \le |x|$

alors $|f(x,y) - L| = |f(x,y) - 2| \le |x| \to 0$ si $x \to 0$

8. $f(x,y) = \dfrac{\cos xy}{1+y^2} \qquad M_0(\pi,1)$

f est définie au point M_0 donc : $\lim\limits_{(x,y)\to(\pi,1)} \dfrac{\cos xy}{1+y^2} = \dfrac{\cos\pi}{1+1} = -\dfrac{1}{2}$

$f(x,y) = \dfrac{xy-2x-y+2}{x^2+y^2-2x-4y+5} \qquad M_0(1,2)$

(I) sur le trajet $x = 1$ on a : $f(1,y) = \dfrac{y-2-y+2}{1+y^2-2-4y+5} = 0$

(II) Sur le trajet $y = 2$: on a : $f(x,2) = \dfrac{2x-2x-2+2}{x^2+4-2x-8+5} = 0$

(III) Sur $y = x+1$ on a :

$f(x,1+x) = \dfrac{x(1+x)-2x-1-x+2}{x^2+(1+x)^2-2x-4-4x+5} = \dfrac{x^2-2x-1}{2x^2-4x-2} = \dfrac{1}{2}$

sur ce trajet la limite est $\dfrac{1}{2}$ alors que $f(x,y)$ n'a pas une limite au point $M_0(1,2)$

Exercice 4

Calculer toutes les dérivées partielles d'ordre 1 des fonctions données :

1. $f(x,y) = y^5 - 4x^2y$
2. $g(x,y) = x^3 + y^2 - 6xy$
3. $h(x,y) = x^2\cos\left(xy^2\right)$
4. $k(x,y) = x^y$
5. $f(x,y,z) = xyz$
6. $g(x,y,z) = x^2\sin zy$
7. $h(x,y,z) = \dfrac{x}{x^2+y^2}e^{-5z}$
8. $k(x,y,z,t) = \sqrt{x^2+y^2}\cos(\omega t - kz)$
9. $m(x,y,z,t) = \dfrac{t}{\sqrt{x^2+y^2+z^2}}$

Solution 4 :

1. $\dfrac{\partial f}{\partial x} = -8xy \qquad \dfrac{\partial f}{\partial y} = 5y^4 - 4x^2$

2. $\dfrac{\partial g}{\partial x} = 3x^2 - 6y \qquad \dfrac{\partial g}{\partial y} = 2y - 6x$
3. $\dfrac{\partial h}{\partial x} = 2x\cos\left(xy^2\right) - x^2y^2\sin\left(xy^2\right) \qquad \dfrac{\partial h}{\partial y} = -2x^3y\sin\left(xy^2\right)$
4. $k = x^y \Longrightarrow \ln k = y\ln x$
$\dfrac{\partial \ln k}{\partial x} = \dfrac{\partial k/\partial x}{k} = \dfrac{y}{x} \Longrightarrow \dfrac{\partial k}{\partial x} = k\dfrac{y}{x} = \dfrac{yx^y}{x} = yx^{y-1}$
$\dfrac{\partial \ln k}{\partial y} = \dfrac{\partial k/\partial y}{k} = \ln x \Longrightarrow \dfrac{\partial k}{\partial y} = k\ln x = x^y\ln x$
5. $\dfrac{\partial f}{\partial x} = yz \qquad \dfrac{\partial f}{\partial y} = xz \qquad \dfrac{\partial f}{\partial z} = xy$
6. $\dfrac{\partial g}{\partial x} = 2x\sin yz \qquad \dfrac{\partial g}{\partial y} = x^2z\cos yz \qquad \dfrac{\partial g}{\partial z} = x^2y\cos yz$
7. $\dfrac{\partial h}{\partial x} = -\dfrac{x^2-y^2}{\left(x^2+y^2\right)^2}e^{-5z} \qquad \dfrac{\partial h}{\partial y} = -\dfrac{2xye^{-5z}}{\left(x^2+y^2\right)^2} \qquad \dfrac{\partial h}{\partial z} = -\dfrac{5xe^{-5z}}{x^2+y^2}$
8. $\dfrac{\partial k}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}\cos\left(\omega t - kz\right) \qquad \dfrac{\partial k}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}\cos\left(\omega t - kz\right)$
$\dfrac{\partial k}{\partial z} = k\sqrt{x^2+y^2}\sin\left(\omega t - kz\right) \qquad \dfrac{\partial k}{\partial t} = -\omega\sqrt{x^2+y^2}\sin\left(\omega t - kz\right)$
9. $\dfrac{\partial m}{\partial x} = \dfrac{-xt}{\left(x^2+y^2+z^2\right)^{3/2}} \qquad \dfrac{\partial m}{\partial y} = \dfrac{-yt}{\left(x^2+y^2+z^2\right)^{3/2}}$
$\dfrac{\partial m}{\partial z} = \dfrac{-zt}{\left(x^2+y^2+z^2\right)^{3/2}} \qquad \dfrac{\partial m}{\partial t} = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

Exercice 5

1 mole de gaz parfait qui occupe le volume V à la température T et à la pression P vérifie la relation $PV = \mathcal{R}T$ où $\mathcal{R}$ est la constante molaire. Montrer que

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1 \text{ et } T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = \mathcal{R}$$

Solution 5 :

$P = \dfrac{\mathcal{R}T}{V} \Longrightarrow \dfrac{\partial P}{\partial V} = -\dfrac{\mathcal{R}T}{V^2}$ et $\dfrac{\partial P}{\partial T} = \dfrac{\mathcal{R}}{V}$

$V = \dfrac{\mathcal{R}T}{P} \Longrightarrow \dfrac{\partial V}{\partial T} = \dfrac{\mathcal{R}}{P}$

$T = \dfrac{PV}{\mathcal{R}} \Longrightarrow \dfrac{\partial T}{\partial P} = \dfrac{V}{\mathcal{R}}$

▲ $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = \left(-\dfrac{\mathcal{R}T}{V^2}\right)\left(\dfrac{\mathcal{R}}{P}\right)\left(\dfrac{V}{\mathcal{R}}\right) = -\dfrac{\mathcal{R}T}{PV} = -1$

▲ $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = T\left(\dfrac{\mathcal{R}}{V}\right)\left(\dfrac{\mathcal{R}}{P}\right) = \dfrac{\mathcal{R}^2T}{PV} = \mathcal{R}$

Exercice 6

La loi de VAN DER WAALS pour n moles d'un gaz s'écrit :

$$\left(P+\frac{n^2a}{V^2}\right)(V-nb)=nRT$$

où P est la pression, V le volume, T la température du gaz, R la constant universelle des gaz et a et b deux constantes positives. Calculer $\frac{\partial T}{\partial P}$ et $\frac{\partial P}{\partial V}$

Solution 6 :

1. $T=\frac{1}{nR}\left(P(V-nb)+\frac{n^2a}{V^2}(V-nb)\right)\Longrightarrow\frac{\partial T}{\partial P}=\frac{V-nb}{nR}$
2. $P+\frac{n^2a}{V^2}=\frac{nRT}{V-nb}\Longrightarrow P=\frac{nRT}{V-nb}-\frac{n^2a}{V^2}$
   $\frac{\partial P}{\partial V}=\frac{\partial}{\partial V}\left(\frac{nRT}{V-nb}\right)-\frac{\partial}{\partial V}\left(\frac{n^2a}{V^2}\right)=\frac{2an^2}{V^3}-\frac{nRT}{(V-bn)^2}$

Exercice 7

Une étude des glaciers a montré que la température T à l'instant t (mesuré en jours) à la profondeur x (mesurée en pied) peut être modélisée par la fonction

$$T(x,t)=T_0+T_1e^{-\lambda x}\sin(\omega t-\lambda x)$$

où $\omega=\frac{2\pi}{365}$ et λ¸sont deux constantes positives.

1. Calculer $\frac{\partial T}{\partial t}$ et $\frac{\partial T}{\partial x}$
2. Montrer que T satisfait l'équation de la chaleur $\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2}$ pour une certaine constante k.

Solution 7 :

1. $\frac{\partial T}{\partial t}=\omega T_1e^{-\lambda x}\cos(\omega t-\lambda x)$
   $\frac{\partial T}{\partial x}=-\lambda T_1e^{-\lambda x}(\cos(\omega t-\lambda x)+\sin(\omega t-\lambda x))$
2. On a
   $\frac{\partial^2 T}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)=2\lambda^2T_1e^{-\lambda x}\cos(\omega t-\lambda x)$
   $=\frac{2\lambda^2}{\omega}\frac{\partial T}{\partial t}$
   Soit donc
   $$\frac{\partial T}{\partial t}=\frac{\omega}{2\lambda^2}\frac{\partial^2 T}{\partial x^2}$$

Exercice 8

Soit $f = f(x,y)$ une fonction à deux variables réelles, définie, continue et admet des dérivées partielles dans un domaine D.
En utilisant les coordonnées polaires (r,θ) ; $x = r\cos\theta$ et $y = r\sin\theta$, montrer que

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

Solution 8 :
On a

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\dfrac{y}{x} \end{cases}$$

Soit $g = \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial f}{\partial \theta}\dfrac{\partial \theta}{\partial x}$

$r = \sqrt{x^2+y^2} \Longrightarrow \dfrac{\partial r}{\partial x} = \dfrac{1}{2}(2x)\left(x^2+y^2\right)^{-1/2} = \dfrac{x}{r} = \cos\theta$

$\theta = \arctan\dfrac{y}{x} \Longrightarrow \dfrac{\partial \theta}{\partial x} = \dfrac{-y/x^2}{1+(y/x)^2} = -\dfrac{y}{x^2+y^2} = -\dfrac{y}{r^2} = -\dfrac{\sin\theta}{r}$

$$g = \frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial g}{\partial \theta}\dfrac{\partial \theta}{\partial x}$

$$\frac{\partial^2 f}{\partial x^2} = \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}$$

$\dfrac{\partial g}{\partial r} = \dfrac{\partial}{\partial r}\left(\cos\theta\dfrac{\partial f}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial f}{\partial \theta}\right) = \cos\theta\dfrac{\partial^2 f}{\partial r^2} - \dfrac{\sin\theta}{r}\dfrac{\partial^2 f}{\partial \theta\partial r} + \dfrac{\sin\theta}{r^2}\dfrac{\partial f}{\partial \theta}$

$\dfrac{\partial g}{\partial \theta} = \dfrac{\partial}{\partial \theta}\left(\cos\theta\dfrac{\partial f}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial f}{\partial \theta}\right) = -\sin\theta\dfrac{\partial f}{\partial r} + \cos\theta\dfrac{\partial^2 f}{\partial r\partial \theta} - \dfrac{\cos\theta}{r}\dfrac{\partial f}{\partial \theta} - \dfrac{\sin\theta}{r}\dfrac{\partial^2 f}{\partial \theta^2}$

$\dfrac{\partial^2 f}{\partial x^2} = \cos\theta\left(\cos\theta\dfrac{\partial^2 f}{\partial r^2} - \dfrac{\sin\theta}{r}\dfrac{\partial^2 f}{\partial \theta\partial r} + \dfrac{\sin\theta}{r^2}\dfrac{\partial f}{\partial \theta}\right)$
$\quad -\dfrac{\sin\theta}{r}\left(-\sin\theta\dfrac{\partial f}{\partial r} + \cos\theta\dfrac{\partial^2 f}{\partial r\partial \theta} - \dfrac{\cos\theta}{r}\dfrac{\partial f}{\partial \theta} - \dfrac{\sin\theta}{r}\dfrac{\partial^2 f}{\partial \theta^2}\right)$

$= \cos^2\theta\dfrac{\partial^2 f}{\partial r^2} - \dfrac{\sin\theta\cos\theta}{r}\dfrac{\partial^2 f}{\partial \theta\partial r} + \dfrac{\sin\theta\cos\theta}{r^2}\dfrac{\partial f}{\partial \theta} + \dfrac{\sin^2\theta}{r}\dfrac{\partial f}{\partial r} - \dfrac{\sin\theta\cos\theta}{r}\dfrac{\partial^2 f}{\partial r\partial \theta} + \dfrac{\sin\theta\cos\theta}{r^2}\dfrac{\partial f}{\partial \theta} + \dfrac{\sin^2\theta}{r^2}\dfrac{\partial^2 f}{\partial \theta^2}$

Soit :

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial \theta\partial r} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r}$$

Posons $h = \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial f}{\partial \theta}\dfrac{\partial \theta}{\partial y}$

$r = \sqrt{x^2+y^2} \Longrightarrow \dfrac{\partial r}{\partial y} = \dfrac{1}{2}(2y)\left(x^2+y^2\right)^{-1/2} = \dfrac{y}{r} = \sin\theta$

$\theta = \arctan \dfrac{y}{x} \Longrightarrow \dfrac{\partial \theta}{\partial y} = \dfrac{1/x}{1+(y/x)^2} = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$

$$h = \frac{\partial f}{\partial y} = \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}$$

$\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial y}{\partial y} = \dfrac{\partial h}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial h}{\partial \theta}\dfrac{\partial \theta}{\partial y}$

$$\frac{\partial^2 f}{\partial y^2} = \sin\theta \frac{\partial h}{\partial r} + \frac{\cos\theta}{r}\frac{\partial h}{\partial \theta}$$

$\dfrac{\partial h}{\partial r} = \dfrac{\partial}{\partial r}\left(\sin\theta \dfrac{\partial f}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial f}{\partial \theta}\right) = \sin\theta \dfrac{\partial^2 f}{\partial r^2} + \dfrac{\cos\theta}{r}\dfrac{\partial^2 f}{\partial \theta \partial r} - \dfrac{\cos\theta}{r^2}\dfrac{\partial f}{\partial \theta}$

$\dfrac{\partial h}{\partial \theta} = \dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial f}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial f}{\partial \theta}\right) = \cos\theta \dfrac{\partial f}{\partial r} + \sin\theta \dfrac{\partial^2 f}{\partial r \partial \theta} + \dfrac{\cos\theta}{r}\dfrac{\partial^2 f}{\partial \theta^2} - \dfrac{\sin\theta}{r}\dfrac{\partial f}{\partial \theta}$

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \sin\theta\left(\sin\theta \frac{\partial^2 f}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2 f}{\partial \theta \partial r} - \frac{\cos\theta}{r^2}\frac{\partial f}{\partial \theta}\right) \\ &\quad + \frac{\cos\theta}{r}\left(\cos\theta \frac{\partial f}{\partial r} + \sin\theta \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\cos\theta}{r}\frac{\partial^2 f}{\partial \theta^2} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right)\end{aligned}$$

$= \sin^2\theta \dfrac{\partial^2 f}{\partial r^2} + \dfrac{\sin\theta\cos\theta}{r}\dfrac{\partial^2 f}{\partial \theta \partial r} - \dfrac{\sin\theta\cos\theta}{r^2}\dfrac{\partial f}{\partial \theta} + \dfrac{\cos^2\theta}{r}\dfrac{\partial f}{\partial r} + \dfrac{\sin\theta\cos\theta}{r}\dfrac{\partial^2 f}{\partial r \partial \theta} + \dfrac{\cos^2\theta}{r^2}\dfrac{\partial^2 f}{\partial \theta^2} - \dfrac{\cos\theta\sin\theta}{r^2}\dfrac{\partial f}{\partial \theta}$

Soit :

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\theta \frac{\partial^2 f}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial \theta \partial r} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r}$$

Substituons :

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \cos^2\theta \frac{\partial^2 f}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial \theta \partial r} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} \\ &\quad + \sin^2\theta \frac{\partial^2 f}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial \theta \partial r} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r}\end{aligned}$$

Finalement :

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}$$

Exercice 9

Vérifier que ω est une différentielle totale. Trouver la fonction f; $\omega = df$:

1. $\omega = \dfrac{ydx - xdy}{x^2+y^2}$
2. $\omega = -y\sin xydx - x\sin xydy$
3. $\omega = 2xy^2dx + \left(2x^2y - z\right)dy + (2z-y)\,dz$
4. $\omega = \dfrac{xdx + ydy + zdz}{\sqrt{x^2+y^2+z^2}}$

Solution 9 :

1. $\omega = \dfrac{ydx}{x^2+y^2} - \dfrac{xdy}{x^2+y^2}$

   $\dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right) = \dfrac{x^2-y^2}{\left(x^2+y^2\right)^2}$ $\qquad$ $\dfrac{\partial}{\partial x}\left(\dfrac{-x}{x^2+y^2}\right) = \dfrac{x^2-y^2}{\left(x^2+y^2\right)^2}$

   ω est une différentielle totale d'une fonction $f(M)$; $\omega = df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

   $\dfrac{\partial f}{\partial x}dx = \dfrac{ydx}{x^2+y^2} \Longrightarrow f(M) = \displaystyle\int \dfrac{ydx}{x^2+y^2} = \int \dfrac{dx}{y\left((x/y)^2+1\right)} = \arctan\dfrac{x}{y} + g(y)$

   $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left(\arctan\dfrac{x}{y} + g(y)\right) = -\dfrac{x}{x^2+y^2} + \dfrac{\partial g}{\partial y} = -\dfrac{x}{x^2+y^2}$

   $\Longrightarrow \dfrac{\partial g}{\partial y} = 0 \Longrightarrow g = C$

   $$f(x,y) = \arctan\frac{x}{y} + C$$

2. $\omega = (-y\sin xy)\,dx - (x\sin xy)\,dy$

   $\dfrac{\partial}{\partial y}(-y\sin xy) = -\sin xy - xy\cos xy$ et $\dfrac{\partial}{\partial x}(-x\sin xy) = -\sin xy - xy\cos xy$

   $\omega = df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

   $\dfrac{\partial f}{\partial x} = -y\sin xy \Longrightarrow f(M) = -\displaystyle\int y\sin xydx = \cos xy + g(y)$

   $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}(\cos xy + g(y)) = -x\sin xy + \dfrac{\partial g}{\partial y} = -x\sin xy \Longrightarrow \dfrac{\partial g}{\partial y} = 0 \Longrightarrow g = C$

   $$f(x,y) = \cos(xy) + C$$

3. $\omega = 2xy^2dx + \left(2x^2y - z\right)dy + (2z-y)\,dz$

   $$\left.\begin{array}{lcl} \dfrac{\partial}{\partial y}\left(2xy^2\right) = 4xy & = & \dfrac{\partial}{\partial x}\left(2x^2y - z\right) = 4xy \\ \dfrac{\partial}{\partial z}\left(2xy^2\right) = 0 & = & \dfrac{\partial}{\partial x}(2z-y) = 0 \\ \dfrac{\partial}{\partial z}\left(2x^2y - z\right) = -1 & = & \dfrac{\partial}{\partial y}(2z-y) = -1 \end{array}\right\}$$

   $\Longrightarrow \omega = df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z}dz$

   $\dfrac{\partial f}{\partial x}dx = 2xy^2dx \Longrightarrow f(M) = \displaystyle\int 2xy^2dx = x^2y^2 + g(y,z)$

   $\dfrac{\partial f}{\partial y} = 2x^2y + \dfrac{\partial g}{\partial y} = 2x^2y - z \Longrightarrow \dfrac{\partial g}{\partial y} = -z \Longrightarrow g(y,z) = -\displaystyle\int zdy = -yz + \varphi(z)$

   $f(M) = x^2y^2 - yz + \varphi(z)$

   $\dfrac{\partial f}{\partial z} = -y + \dfrac{d\varphi}{dz} = 2z - y \Longrightarrow \dfrac{d\varphi}{dz} = 2z \Longrightarrow \varphi(z) = \displaystyle\int 2zdz = z^2 + C$

   $$f(x,y,z) = x^2y^2 - yz + z^2 + C$$

4. $\omega = \dfrac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}}$

Posons $r = \sqrt{x^2 + y^2 + z^2} \Longrightarrow \omega = \dfrac{x}{r}dx + \dfrac{y}{r}dy + \dfrac{z}{r}dz$

$$\frac{\partial}{\partial y}\left(\frac{x}{r}\right) = x\frac{\partial\left(1/r\right)}{\partial y} = x\frac{\partial\left(x^2 + y^2 + z^2\right)^{-1/2}}{\partial y}$$

$$= x\left(-\frac{1}{2}\left(\left(x^2 + y^2 + z^2\right)^{-3/2}\right) \times 2y\right) = -x\frac{y}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} = \frac{-xy}{\sqrt{r^3}}$$

Remarquons que les variables x, y, et z figurent de la même manière dans r donc on a :

$$\frac{\partial\left(1/r\right)}{\partial \alpha} = \frac{-\alpha}{\sqrt{r^3}}; \alpha = x, y, z$$

D'où :

$$\left.\begin{array}{lllll}
\dfrac{\partial}{\partial y}\left(\dfrac{x}{r}\right) = \dfrac{-xy}{\sqrt{r^3}} & = & \dfrac{\partial}{\partial x}\left(\dfrac{y}{r}\right) = \dfrac{-xy}{\sqrt{r^3}} \\
\dfrac{\partial}{\partial z}\left(\dfrac{x}{r}\right) = \dfrac{-xz}{\sqrt{r^3}} & = & \dfrac{\partial}{\partial x}\left(\dfrac{z}{r}\right) = \dfrac{-xz}{\sqrt{r^3}} \\
\dfrac{\partial}{\partial z}\left(\dfrac{y}{r}\right) = \dfrac{-zy}{\sqrt{r^3}} & = & \dfrac{\partial}{\partial y}\left(\dfrac{z}{r}\right) = \dfrac{-zy}{\sqrt{r^3}}
\end{array}\right\}$$

$$\Longrightarrow \omega = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \Longrightarrow f\left(M\right) = \int \frac{xdx}{\sqrt{x^2 + y^2 + z^2}}$$

posons $t = x^2 + y^2 + z^2 \Longrightarrow dt = 2xdx$

$$f\left(M\right) = \int \frac{1}{2}\frac{dt}{\sqrt{t}} = \sqrt{t} = \sqrt{x^2 + y^2 + z^2} + g\left(y, z\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{1/2} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} + \frac{\partial g}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\Longrightarrow \frac{\partial g}{\partial y} = 0 \Longrightarrow g = g\left(z\right)$$

$$f\left(M\right) = \sqrt{x^2 + y^2 + z^2} + g\left(z\right)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{1/2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} + \frac{dg}{dz} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\Longrightarrow \frac{dg}{dz} = 0 \Longrightarrow g = C$$

$$f\left(M\right) = \sqrt{x^2 + y^2 + z^2} + C$$

Exercice 10

Montrer que :

$$\omega = \left(y^2 + ze^y + yz^2\right)dx + \left(2xy + xze^y + xz^2\right)dy + \left(xe^y + 2xyz\right)dz$$

est une différentielle totale et intégrer ω.

Solution 10 On a :

$$\begin{cases} \dfrac{\partial P}{\partial y} = 2y + ze^y + z^2 = \dfrac{\partial Q}{\partial x} \\ \dfrac{\partial Q}{\partial z} = xe^y + 2xz = \dfrac{\partial R}{\partial x} \\ \dfrac{\partial R}{\partial x} = e^y + 2yz = \dfrac{\partial P}{\partial z} \end{cases}$$

$\Longrightarrow \omega$ est une différentielle totale par suite il existe $\varphi = \varphi(x, y, z)$ telle que $\omega = d\varphi$

$\dfrac{\partial \varphi}{\partial x} = y^2 + ze^y + yz^2$

$\Longrightarrow \varphi = \int \left(y^2 + ze^y + yz^2\right) dx = xy^2 + xyz^2 + xze^y + f(y, z)$

$\dfrac{\partial \varphi}{\partial y} = 2xy + xz^2 + xze^y + \dfrac{\partial f}{\partial y} = 2xy + xze^y + xz^2$

$\Longrightarrow \dfrac{\partial f}{\partial y} = 0 \Longrightarrow f = g(z)$

$\varphi = xy^2 + xyz^2 + xze^y + g(z)$

$\dfrac{\partial \varphi}{\partial z} = 2xyz + xe^y + g' = xe^y + 2xyz \Longrightarrow g' = 0$

$\Longrightarrow g = C$

$$\varphi = xy^2 + xyz^2 + xze^y + C$$

Exercice 11

Montrer que $\omega = \cos x \cos y dx - \sin x \sin y dy$ est une différentielle totale et déterminer la solution de l'équation diférentielle :

$$y' \tan x = \cot y$$

Solution 11 :

$\dfrac{\partial}{\partial y}(\cos x \cos y) = -\cos x \sin y$ et $\dfrac{\partial}{\partial x}(-\sin x \sin y) = -\cos x \sin y$

$\Longrightarrow \omega = df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

$\dfrac{\partial f}{\partial x} = \cos x \cos y \Longrightarrow f(x, y) = \displaystyle\int \cos x \cos y dx = \cos y \sin x + g(y)$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}(\cos y \sin x + g(y)) = -\sin y \sin x + \dfrac{\partial g}{\partial y} = -\sin y \sin x$

$\Longrightarrow \dfrac{\partial g}{\partial y} = 0 \Longrightarrow g = C$

$$f(x, y) = \cos y \sin x + C$$

$\omega = \cos x \cos y dx - \sin x \sin y dy = 0 \Longleftrightarrow \cos x \cos y dx = \sin x \sin y dy$

$\Longleftrightarrow \dfrac{dy}{dx} \sin x \sin y = \cos x \cos y \Longrightarrow y' \tan x = \cot y$

Alors la solution est tellque $\omega = 0 \Longleftrightarrow f(x, y) = Cte$,

soit $\cos y \sin x = k \Longrightarrow y(x) = \arccos\left(\dfrac{k}{\sin x}\right)$

Pour les valeurs de k telles que $\left|\dfrac{k}{\sin x}\right| \leq 1$

Exercice 12

Vérifier que

$$\omega = y^2 z dx + 2xyzdy - 2xy^2 dz$$

n'est pas une différentielle totale ; trouver un facteur intégrant $\varphi(z)$ tel que $\omega\varphi(z)$ soit une différentielle exacte et intégrer $\omega\varphi(z)$.

Solution 12 $\dfrac{\partial(2xyz)}{\partial z} = 2xy \neq \dfrac{\partial(-2xy^2)}{\partial y} = -4xy$ donc ω n'est pas une différentielle totale

Si $\varphi(z)$ est un facteur intégrant de ω alors $\omega\varphi$ devient une différentielle totale c'est-à-dire :
$\omega\varphi = y^2 z\varphi dx + 2xyz\varphi dy - 2xy^2\varphi dz = df$ par suite on aura :

$$\frac{\partial(y^2 z\varphi)}{\partial z} = \frac{\partial(-2xy^2\varphi)}{\partial x} \Longrightarrow y^2\left(\varphi + z\frac{d\varphi}{dz}\right) = -2y^2\varphi$$

$$\Longrightarrow \varphi + z\frac{d\varphi}{dz} = -2\varphi \Longrightarrow z\frac{d\varphi}{dz} + 3\varphi = 0$$

$$\frac{\partial(2xyz\varphi)}{\partial z} = \frac{\partial(-2xy^2\varphi)}{\partial y} \Longrightarrow 2xy\left(\varphi + z\frac{d\varphi}{dz}\right) = -2x(2y\varphi)$$

$$\varphi + z\frac{d\varphi}{dz} = -2\varphi \Longrightarrow z\frac{d\varphi}{dz} + 3\varphi = 0$$

$\varphi(z)$ est la solution de l'équation différentielle $z\dfrac{d\varphi}{dz} + 3\varphi = 0$

$$z\frac{d\varphi}{dz} + 3\varphi = 0 \Longrightarrow z\frac{d\varphi}{dz} = -3\varphi \Longrightarrow \frac{d\varphi}{\varphi} = -3\frac{dz}{z}$$

l'intégration nous donne : $\ln|\varphi| = -3\ln|z| + c$ soit $\varphi(z) = \dfrac{k}{z^3}$

$$\omega\varphi = k\frac{y^2 zdx + 2xyzdy - 2xy^2 dz}{z^3} = k\frac{y^2}{z^2}dx + \frac{2kxy}{z^2}dy - \frac{2kxy}{z^3}dz = df$$

$$\frac{\partial f}{\partial x} = k\frac{y^2}{z^2} \Longrightarrow f(x,y,z) = k\int\frac{y^2}{z^2}dx = k\frac{xy^2}{z^2} + g(y,z)$$

$$\frac{\partial f}{\partial y} = k\frac{2xy}{z^2} + \frac{\partial g}{\partial y} = k\frac{2xy}{z^2} \Longrightarrow \frac{\partial g}{\partial y} = 0 \text{ et } g = g(z) \quad \text{c-à-d } f(M) = k\frac{xy^2}{z^2} + g(z)$$

$$\frac{\partial f}{\partial z} = -\frac{2kxy}{z^3} + \frac{dg}{dz} = -\frac{2kxy}{z^3} \Longrightarrow \frac{dg}{dz} = 0 \text{ et } g = C$$

$$f(M) = k\frac{xy^2}{z^2} + C$$

Exercice 13

Trouver un facteur intégrant $\varphi(x)$ pour la forme différentielle :

$$\omega = \left(\frac{1}{x+y} - \ln(x+y)\right)dx + \frac{dy}{x+y}$$

et intégrer $\omega\varphi(x)$.

Solution 13 :
$\omega\varphi = \left(\dfrac{1}{x+y} - \ln(x+y)\right)\varphi dx + \dfrac{\varphi dy}{x+y}$ est une différentielle totale si :
$\dfrac{\partial}{\partial y}\left(\dfrac{\varphi}{x+y} - \varphi\ln(x+y)\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\varphi}{x+y}\right)$ c'est-à-dire
$\dfrac{-\varphi}{(x+y)^2} - \dfrac{\varphi}{x+y} = \dfrac{\varphi'(x+y) - \varphi}{(x+y)^2} \Longrightarrow -\dfrac{\varphi}{x+y} = \dfrac{\varphi'(x+y)}{(x+y)^2} = \dfrac{\varphi'}{x+y}$
donc pour $\varphi' = \dfrac{d\varphi}{dx} = -\varphi \Longrightarrow \dfrac{d\varphi}{\varphi} = -dx$
$\Longrightarrow \ln|\varphi| = -x + C$ ou bien $\varphi(x) = ke^{-x}$
$\omega\varphi = \left(\dfrac{ke^{-x}}{x+y} - ke^{-x}\ln(x+y)\right)dx + \dfrac{ke^{-x}dy}{x+y} = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$
$\dfrac{\partial f}{\partial y} = \dfrac{ke^{-x}}{x+y} \Longrightarrow k\displaystyle\int \dfrac{e^{-x}dy}{x+y} = ke^{-x}\ln(x+y) + g(x)$
$\dfrac{\partial f}{\partial x} = -ke^{-x}\ln(x+y) + k\dfrac{e^{-x}}{x+y} + \dfrac{dg}{dx} = k\dfrac{e^{-x}}{x+y} - ke^{-x}\ln(x+y)$
$\Longrightarrow \dfrac{dg}{dx} = 0 \Longrightarrow g = C$

$$f(x,y) = ke^{-x}\ln(x+y) + C$$

Exercice 14
On considère la forme différentielle

$$\omega = 2xy\sin y dx + x^2(\sin y + y\cos y)\,dy - \frac{3x^2 y\sin y}{z}dz$$

1. Montrer que ω n'est pas une différentielle totale.
2. Trouver un facteur intégrant $\mu = \mu(z)$ tel que $\mu\omega$ soit une différentielle totale.
3. Déterminer la fonction $f = f(x,y,z)$ telle que $\mu\omega = df$.

Solution 14 : Soit $\omega = Pdx + Qdy + Rdz$

1. on a $\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}(2xy\sin y) = 2x(\sin y + y\cos y) = \dfrac{\partial Q}{\partial x}$
   $\dfrac{\partial P}{\partial z} = 0 \neq \dfrac{\partial R}{\partial x} = -\dfrac{6xy\sin y}{z}$
   Donc ω n'est pas une différentielle totale.
2. Si $\mu = \mu(z)$ est un facteur intégrant donc $\mu\omega$ une différentielle totale
   par suite on aura $\dfrac{\partial \mu P}{\partial z} = \dfrac{\partial \mu R}{\partial x}$
   $\dfrac{\partial}{\partial z}(2\mu xy\sin y) = \dfrac{\partial}{\partial x}\left(-\dfrac{3\mu x^2 y\sin y}{z}\right)$
   $\Longrightarrow 2\dfrac{d\mu}{dz}xy\sin y = -\dfrac{6\mu xy\sin y}{z}$
   $\Longrightarrow \dfrac{d\mu}{dz} = -\dfrac{3\mu}{z}$

$\Longrightarrow \frac{d\mu}{\mu} = -3\frac{dz}{z}$

$\ln \mu = -3 \ln z \Longrightarrow \mu = \frac{1}{z^3}$

3. $\mu\omega = \frac{2xy \sin y}{z^3} dx + \frac{x^2}{z^3} (\sin y + y \cos y)\, dy - \frac{3x^2 y \sin y}{z^4} dz = df$

$f = \int \frac{2xy \sin y}{z^3} dx = \frac{x^2 y}{z^3} \sin y + g(y, z)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2 y}{z^3} \sin y\right) + \frac{\partial g}{\partial y}$

$= \frac{x^2}{z^3} (\sin y + y \cos y) + \frac{\partial g}{\partial y} = \mu Q = \frac{x^2}{z^3} (\sin y + y \cos y)$

$\Longrightarrow \frac{\partial g}{\partial y} = 0 \Longrightarrow g = g(z)$

$f = \frac{x^2 y}{z^3} \sin y + g(z)$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x^2 y}{z^3} \sin y\right) + g' = -\frac{3x^2 y}{z^4} \sin y + g' = \mu R$

$g' = 0 \Longrightarrow g = C$

$f(x, y, z) = \frac{x^2 y}{z^3} \sin y + C$

$f(0, 0, 1) = 1 \Longrightarrow C = 1$

$$f(x, y, z) = \frac{x^2 y}{z^3} \sin y + 1$$

Exercice 15

Montrer que la forme différentielle $\omega = ydx + (2x - ye^y)\, dy$ n'est pas exacte dans $\mathbb{R}^2$. Trouver une fonction $\mu(y)$ telle que la forme différentielle $\varphi(x, y) = \mu\omega$ soit exacte dans $\mathbb{R}^2$. Déterminer ensuite les potentiels $f(x, y)$ de $\varphi./\varphi = df$

Solution 15 :

On a $\frac{\partial}{\partial y}(y) = 1 \neq \frac{\partial(2x - ye^y)}{\partial x} = 2$ donc ω n'est pas exacte dans $\mathbb{R}^2$

Soit $\mu(y)$ le facteur intégrant de ω : on aura donc

$\frac{\partial}{\partial y}(\mu y) = \frac{\partial(2\mu x - y\mu e^y)}{\partial x} \Longleftrightarrow \mu + y\mu' = 2\mu$

$\Longleftrightarrow y\frac{d\mu}{dy} = \mu \Longleftrightarrow \frac{d\mu}{\mu} = \frac{dy}{y}$ soit $\mu(y) = ay$ où a est une constante arbitraires

$\Longrightarrow \mu\omega = ay^2 dx + a(2xy - y^2 e^y)\, dy = df$

$\frac{\partial f}{\partial x} = ay^2 \Longrightarrow f(x, y) = axy^2 + g(y)$

$\frac{\partial f}{\partial y} = 2axy + g'(y) = a(2xy - y^2 e^y) \Longrightarrow g' = -ay^2 e^y$

$g(y) = \int ay^2 e^y dy = -a(y^2 - 2y + 2)\, e^y + C$ soit donc

$$f(x, y) = axy^2 - a(y^2 - 2y + 2)\, e^y + C$$

Exercice 16

Déterminer la valeur du paramètre μ de sorte que la forme différentielle

$$\omega = \left(x^2 - 5\mu y + 8yz\right) dx + \left(-5x + 8\mu xz + 2\right) dy + \left(\left(7+\mu\right) xy - 6z\right) dz$$

soit exacte.
Pour cette valeur de μ déterminer la primitive φ de ω telle que $\varphi\left(3,1,-2\right) = 0$

Solution 16 :

ω est une différentielle totale si

$$\begin{cases} \dfrac{\partial}{\partial y}\left(x^2 - 5\mu y + 8yz\right) = \dfrac{\partial}{\partial x}\left(-5x + 8\mu xz + 2\right) \\ \dfrac{\partial}{\partial x}\left(\left(7+\mu\right) xy - 6z\right) = \dfrac{\partial}{\partial z}\left(x^2 - 5\mu y + 8yz\right) \\ \dfrac{\partial}{\partial y}\left(\left(7+\mu\right) xy - 6z\right) = \dfrac{\partial}{\partial z}\left(-5x + 8\mu xz + 2\right) \end{cases}$$

$$\Longleftrightarrow \begin{cases} -5\mu + 8z = -5 + 8\mu z \\ \left(7+\mu\right) y = 8y \\ \left(\mu+7\right) x = 8\mu x \end{cases} \Longrightarrow \mu = 1$$

Alors :

$$\omega = \left(x^2 - 5y + 8yz\right) dx + \left(-5x + 8xz + 2\right) dy + \left(8xy - 6z\right) dz$$

ω est une différentielle totale alors $\exists\ \varphi / \omega = d\varphi$.

donc : $\dfrac{\partial \varphi}{\partial x} = x^2 - 5y + 8yz \Longrightarrow \varphi = \dfrac{x^3}{3} - 5xy + 8xyz + g\left(y,z\right)$

$\dfrac{\partial \varphi}{\partial y} = -5x + 8xz + \dfrac{\partial g}{\partial y} = -5x + 8xz + 2 \Longrightarrow \dfrac{\partial g}{\partial y} = 2$

$g = 2y + h\left(z\right) \Longrightarrow \varphi = \dfrac{x^3}{3} - 5xy + 8xyz + 2y + h\left(z\right)$

$\dfrac{\partial \varphi}{\partial z} = 8xy + h' = 8xy - 6z$

$h' = -6z \Longrightarrow h = -3z^2 + C$

$$\varphi\left(x,y,z\right) = \frac{x^3}{3} - 5xy + 8xyz + 2y - 3z^2 + C$$

$\varphi\left(3,1,-2\right) = \dfrac{3^3}{3} - 5\times 3 + 8\times 3\times\left(-2\right) + 2 - 3\times 4 + C = C - 64 = 0 \Longrightarrow C = 64$

$$\varphi\left(x,y,z\right) = \frac{x^3}{3} - 5xy + 8xyz + 2y - 3z^2 + 64$$

Exercice 17

Le volume d'un cône de révolution droit de rayon de base r et de hauteur h est donné par $V = \dfrac{\pi h r^2}{3}$.

1. Les mesures de r et h, avec une incertitude de 0.1 cm, sont : $h = 25$ cm et $r = 10$ cm. Estimer à l'aide de différentielle l'incertitude sur le volume.
2. Refaire le même exercice pour un parallélipipède de dimmensions :

$$x = 10 \pm 0.1\,\text{cm},\ y = 15 \pm 0.2\,\text{cm}\ et\ \ z = 12 \pm 0.1\,\text{cm}$$

Solution 17 :

1. $dV = \frac{2\pi}{3}hrdr + \frac{\pi}{3}r^2dh \Longrightarrow \Delta V = \frac{2\pi}{3}hr\Delta r + \frac{\pi}{3}r^2\Delta h$

   $\Delta V = \frac{2\pi}{3} \times (25 \times 10) \times 0.1 + \frac{\pi}{3} \times (10)^2 \times 0.1 = 20\pi = 62.8$

2. $V = xyz \Longrightarrow dV = yzdx + xzdy + xydz$

   $\Longrightarrow \Delta V = yz\Delta x + xz\Delta y + xy\Delta z$

   $= 15 \times 12 \times 0.1 + 10 \times 12 \times 0.2 + 10 \times 15 \times 0.1 = 57$

Exercice 18

L'équation d'état d'un gaz parfait est $PV = nRT$ où P est la pression du gaz dans le volume V à la température T. n est le nombre des moles et R est la constante universelle des gaz parfaits $R = 8.314510\,\mathrm{J\,mol^{-1}\,K^{-1}}$. Déterminer :

1. La variation de la pression consécutive à une variation de volume dV et de température dT.
2. Refaire le même calcul pour V et T.

Solution 18 :

1. $P = nR\frac{T}{V} \Longrightarrow dP = nR\left(\frac{dT}{V} - \frac{T}{V^2}dV\right)$
2. $V = nR\frac{T}{P} \Longrightarrow dV = nR\left(\frac{dT}{P} - \frac{T}{P^2}dP\right)$

   $T = \frac{PV}{nR} \Longrightarrow dT = \frac{1}{nR}(PdV + VdP)$

Exercice 19

La période d'un pendule simple s'exprime en fonction de la longueur ℓ de l'accélération du pesanteur g par la formule $T = 2\pi\sqrt{\frac{\ell}{g}}$.
Calculer la variation de T en fonctions des variations de ℓ et g.

Solution 19 :

$T = 2\pi\ell^{1/2}g^{-1/2}$

$dT = 2\pi\left(\frac{\partial T}{\partial \ell}d\ell + \frac{\partial T}{\partial g}dg\right) = 2\pi\left(\frac{1}{2}\ell^{-1/2}g^{-1/2}d\ell - \frac{1}{2}\ell^{1/2}g^{-3/2}dg\right)$

$$dT = \pi\left(\sqrt{\frac{1}{g\ell}}d\ell - \sqrt{\frac{\ell}{g^3}}dg\right)$$

Exercice 20

L'intensité du champ magnétique créé par une spire de rayon R sur un point M de son axe z est :

$$B(I,z) = \frac{\mu_0 I}{2} \frac{R^2}{\left(\sqrt{z^2+R^2}\right)^3}$$

Où I est l'intensité du courant électrique traversant la spire et μ_0 est la perméabilité magnétique du vide. On suppose que R est constant.
Calculer, à l'aide de différentielle totale, la variation de B due aux variations de z et I.

Solution 20 : μ_0 et R sont constants.

$$B = \frac{\mu_0 R^2}{2} I \left(z^2+R^2\right)^{-3/2}$$

$$\frac{\partial B}{\partial I} = \frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}}$$

$$\frac{\partial B}{\partial z} = \left(\frac{\mu_0 R^2}{2} I\right)\left(-\frac{3}{2}\right)(2z)\left(z^2+R^2\right)^{-5/2} = -\frac{3\mu_0 I R^2 z}{2\left(z^2+R^2\right)^{5/2}}$$

$$\Delta B = \frac{\partial B}{\partial I}\Delta I + \frac{\partial B}{\partial z}\Delta z = \frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}}\Delta I - \frac{3\mu_0 I R^2 z}{2\left(z^2+R^2\right)^{5/2}}\Delta z$$

$$= \frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}}\left(\Delta I - \frac{3Iz}{z^2+R^2}\Delta z\right)$$

Exercice 21

On considère un champ magnétique dont l'intensité en un point $M(x,y,z)$ est donnée par

$$B(x,y,z) = \frac{x+y+z}{x^2+y^2+z^2}$$

1. Calculer l'erreur de mésure sur B, si les erreurs sur les valeurs de x, y, z sont d'ordre 0.1
2. Quelle est la variation de B en un point $N(x+0.2, y-0.1, z)$

Solution 21 :

1. $dB = \frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy + \frac{\partial B}{\partial z}dz$

$$\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x+y+z}{x^2+y^2+z^2}\right) = \frac{2x(x+y+z)-\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^2} = \frac{2x(x+y+z)-r^2}{r^4}$$

de même $\frac{\partial B}{\partial y} = \frac{2y(x+y+z)-r^2}{r^4}$

et $\frac{\partial B}{\partial z} = \frac{2z(x+y+z)-r^2}{r^4}$

$$dB = \frac{2x(x+y+z)-r^2}{r^4}dx + \frac{2y(x+y+z)-r^2}{r^4}dy + \frac{2z(x+y+z)-r^2}{r^4}dz$$

$$\Delta B = \frac{2x(x+y+z)-r^2}{r^4}\Delta x + \frac{2y(x+y+z)-r^2}{r^4}\Delta y + \frac{2z(x+y+z)-r^2}{r^4}\Delta z$$

$$= \left(\frac{2x(x+y+z)-r^2}{r^4} + \frac{2y(x+y+z)-r^2}{r^4} + \frac{2z(x+y+z)-r^2}{r^4}\right)(0.1)$$

$$= \left(-\frac{1}{r^2} + \frac{4\,(xy + xz + yz)}{r^4}\right)(0.1)$$

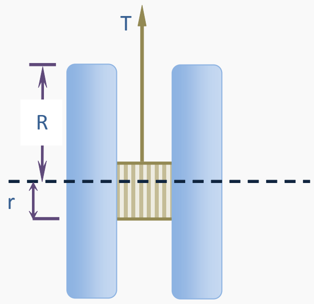
2. $\Delta B = \dfrac{2x\,(x+y+z) - r^2}{r^4}\,(0.2) + \dfrac{2y\,(x+y+z) - r^2}{r^4}\,(-0.1)$

Exercice 22

La tension T de la cordelette du yo-yo en figure est décrite par la fonction

$$T = \frac{mgR}{2r^2 + R^2}$$

où m est lamasse du yo-yo et g l'accélération due à la gravité.
Donner une estimation de la variation de T lorsque R passe de 3 cm à 3.1 cm et r passe de 0.7 cm à 0.8 cm.



Solution 22 :

$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial R}dR$$

$$\frac{\partial T}{\partial r} = -\frac{4mgRr}{(R^2+2r^2)^2} \quad \text{et} \quad \frac{\partial T}{\partial R} = \frac{mg\,(2r^2 - R^2)}{(R^2+2r^2)^2}$$

$$dT = -\frac{4mgRr}{(R^2+2r^2)^2}dr + \frac{mg\,(2r^2 - R^2)}{(R^2+2r^2)^2}dR$$

$$= -4\frac{mg\,(3)\,(0.7)}{\left(9 + 2\,(0.7)^2\right)^2}(0.1) + \frac{mg\left(2\,(0.7)^2 - 9\right)}{\left(9 + 2\,(0.7)^2\right)^2}(0.1) = -1.6486 \times 10^{-2} mg$$

Exercice 23

Déterminer le jacobien de transformation de coordonnées dans les cas suivants :

1. $(u,v) \to (x,y) : u = x + 2y \quad v = x - y$
2. $(x,y) \to (u,v) : \; u = x + 2y \quad v = x - y$
3. $(x,y) \to (\rho,\varphi) : \; x = \rho\cos^2\varphi \quad y = \rho\sin^2\varphi$
4. $(x,y) \to (\theta,\varphi) : x = \dfrac{\sin\theta}{\cos\varphi} \qquad y = \dfrac{\sin\varphi}{\cos\theta}$
5. $(x,y,z) \to (u,v,w) : x = u,\, y = u + v + w,\, z = u - v + w$
6. $(x,y,z) \to (r,\theta,\varphi) : x = ar\cos\theta,\, y = ar\sin\theta$ et $z = br\varphi$
7. $(x,y,z) \to (u,v,w) : x = u + v + w,\; y = uvw \qquad z = \dfrac{1}{u} + \dfrac{1}{v} + \dfrac{1}{w}$

8. $(x, y, z) \to (r, \theta, \varphi) : x = \dfrac{\cos\theta}{\cos\varphi}; y = \dfrac{\sin\theta}{\cos\varphi}; z = \tan\varphi$

Solution 23 1. $(u, v) \to (x, y) : u = x + 2y \quad v = x - y$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -3$$

2. $(x, y) \to (u, v) : \ u = x + 2y \quad v = x - y$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = -\frac{1}{3}$$

ou bien : $u = x + 2y$ et $v = x - y \Longrightarrow$

$u - v = 3y \to y = \dfrac{u-v}{3}$

$u + 2v = 3x \to x = \dfrac{u+2v}{3}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = -\frac{1}{3}$$

3. $(x, y) \to (\rho, \varphi) : \ x = \rho\cos^2\varphi \quad y = \rho\sin^2\varphi$

$$J = \frac{\partial(x,y)}{\partial(\rho,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos^2\varphi & -2\rho\cos\varphi\sin\varphi \\ \sin^2\varphi & 2\rho\cos\varphi\sin\varphi \end{vmatrix}$$

$$= 2\rho\cos^3\varphi\sin\varphi + 2\rho\cos\varphi\sin^3\varphi = 2\rho\cos\varphi\sin\varphi\left(\cos^2\varphi + \sin^2\varphi\right)$$
$$= 2\rho\cos\varphi\sin\varphi = \rho\sin 2\varphi$$

4. $(x, y) \to (\theta, \varphi) : x = \dfrac{\sin\theta}{\cos\varphi} \qquad y = \dfrac{\sin\varphi}{\cos\theta}$

$$J = \frac{\partial(x,y)}{\partial(\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \frac{\cos\theta}{\cos\varphi} & \frac{\sin\theta\sin\varphi}{\cos^2\varphi} \\ \frac{\sin\varphi\sin\theta}{\cos^2\theta} & \frac{\cos\varphi}{\cos\theta} \end{vmatrix}$$

$$= \frac{\cos\theta}{\cos\varphi} \times \frac{\cos\varphi}{\cos\theta} - \frac{\sin\theta\sin\varphi}{\cos^2\varphi} \times \frac{\sin\varphi\sin\theta}{\cos^2\theta} = 1 - \frac{\sin^2\theta\sin^2\varphi}{\cos^2\theta\cos^2\varphi} = 1 - \tan^2\theta\tan^2\varphi$$

5. $(x, y, z) \to (u, v, w) : x = u, y = u + v + w, z = u - v + w$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2$$

6. $(x, y, z) \to (r, \theta, \varphi) : x = ar\cos\theta, y = ar\sin\theta$ et $z = br\varphi$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta & 0 \\ a\sin\theta & ar\cos\theta & 0 \\ b\varphi & 0 & br \end{vmatrix}$$

$$= ba^2r^2\cos^2\theta + ba^2r^2\sin^2\theta = a^2br^2$$

7. $(x,y,z) \longrightarrow (u,v,w) : x = u+v+w,\ y = uvw \qquad z = \frac{1}{u}+\frac{1}{v}+\frac{1}{w}$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ uw & uw & uv \\ -\frac{1}{u^2} & -\frac{1}{v^2} & -\frac{1}{w^2} \end{vmatrix}$$

$$= \frac{u}{v} - \frac{v}{u} + \frac{w}{u} - \frac{uw}{v^2} = \frac{1}{uv^2}\left(u^2 - v^2\right)(v-w)$$

8. $(x,y,z) \longrightarrow (r,\theta,\varphi) : x = r\frac{\cos\theta}{\cos\varphi}; y = r\frac{\sin\theta}{\cos\varphi}; z = r\tan\varphi$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \frac{\cos\theta}{\cos\varphi} & -r\frac{\sin\theta}{\cos\varphi} & r\frac{\cos\theta\sin\varphi}{\cos^2\varphi} \\ \frac{\sin\theta}{\cos\varphi} & r\frac{\cos\theta}{\cos\varphi} & r\frac{\sin\theta\sin\varphi}{\cos^2\varphi} \\ \tan\varphi & 0 & \frac{r}{\cos^2\varphi} \end{vmatrix}$$

$$= \frac{1}{\cos^4\varphi}\left(r^2\cos^2\theta + r^2\sin^2\theta - r^2\cos^2\theta\cos\varphi\sin\varphi\tan\varphi - r^2\cos\varphi\sin^2\theta\sin\varphi\tan\varphi\right)$$

$$= \frac{r^2}{\cos^4\varphi}\left(1 - \frac{1}{2}\sin 2\varphi\tan\varphi\right)$$