

# Opérateurs hamiltonien

## Exercices corrigés

○ Dans tous les exercices suivants, on désigne par

$$\vec{r} = \overrightarrow{OM} = x \vec{i} + y \vec{j} + z \vec{k}$$

le rayon vecteur d'un point  $M(x, y, z)$  de l'espace rapporté au système d'axes  $Ox, Oy, Oz$  et

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \neq 0$$

### Exercice 1

Trouver le gradient et le laplacien du champ scalaire  $f = f(M)$  dans les cas suivants :

1.  $f(x, y, z) = xy + xz + yz$

2.  $f(x, y, z) = xy^2z^3$

3.  $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

4.  $f(x, y, z) = e^{-xyz}$

5.  $f(x, y, z) = e^{ax+by+cz}$

6.  $f(x, y, z) = (x^2 + y^2 + z^2)^n, n \in \mathbb{Z}$ .



**Solution 1 :** On a  $\vec{\nabla} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$  et  $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

1.  $f(x, y, z) = xy + xz + yz$

$$\frac{\partial f}{\partial x} = y + z \implies \frac{\partial^2 f}{\partial x^2} = 0$$

$$\frac{\partial f}{\partial y} = x + z \implies \frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial f}{\partial z} = x + y \implies \frac{\partial^2 f}{\partial z^2} = 0$$

$$\vec{\nabla} f = (y + z) \vec{i} + (x + z) \vec{j} + (x + y) \vec{k} \quad \text{et} \quad \Delta f = 0$$

2.  $f(x, y, z) = xy^2z^3$

$$\frac{\partial f}{\partial x} = y^2z^3 \quad \frac{\partial f}{\partial y} = 2xyz^3 \quad \frac{\partial f}{\partial z} = 3xy^2z^2$$

$$\vec{\nabla} f = y^2z^3 \vec{i} + 2xyz^3 \vec{j} + 3xy^2z^2 \vec{k}$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial^2 f}{\partial y^2} = 2xz^3 \quad \frac{\partial^2 f}{\partial z^2} = 6xy^2z$$

$$\Delta f = 2xz^3 + 6xy^2z = 2xz(z^2 + 3y^2)$$

3.  $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (\ln(x^2 + y^2 + z^2)) = \frac{2x}{x^2 + y^2 + z^2}$$

de même :  $\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}$  et  $\frac{\partial f}{\partial z} = \frac{2z}{x^2 + y^2 + z^2}$

$$\vec{\nabla} f = \frac{2x \vec{i} + 2y \vec{j} + 2z \vec{k}}{x^2 + y^2 + z^2} = \frac{2 \vec{r}}{r^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \frac{2x}{x^2 + y^2 + z^2} = 2 \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2}$$

de même :  $\frac{\partial^2 f}{\partial y^2} = 2 \frac{x^2 - y^2 + z^2}{(x^2 + y^2 + z^2)^2}$  et  $\frac{\partial^2 f}{\partial z^2} = 2 \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2}$

$$\Delta f = 2 \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} + 2 \frac{x^2 - y^2 + z^2}{(x^2 + y^2 + z^2)^2} + 2 \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{2}{x^2 + y^2 + z^2} = \frac{2}{r^2}$$

4.  $f(x, y, z) = e^{-xyz}$

$$\frac{\partial f}{\partial x} = -yze^{-xyz} \quad \frac{\partial^2 f}{\partial x^2} = y^2 z^2 e^{-xyz}$$

$$\frac{\partial f}{\partial y} = -xze^{-xyz} \quad \frac{\partial^2 f}{\partial x^2} = x^2 z^2 e^{-xyz}$$

$$\frac{\partial f}{\partial z} = -xye^{-xyz} \quad \frac{\partial^2 f}{\partial x^2} = x^2 y^2 e^{-xyz}$$

$$\vec{\nabla} f = - \left( yz \vec{i} + xz \vec{j} + xy \vec{k} \right) e^{-xyz}$$

$$\Delta f = (y^2 z^2 + x^2 z^2 + x^2 y^2) e^{-xyz}$$

5.  $f(x, y, z) = e^{ax+by+cz}$

$$\frac{\partial f}{\partial x} = ae^{ax+by+cz} \quad \frac{\partial^2 f}{\partial x^2} = a^2 e^{ax+by+cz}$$

$$\frac{\partial f}{\partial y} = be^{ax+by+cz} \quad \frac{\partial^2 f}{\partial y^2} = b^2 e^{ax+by+cz}$$

$$\frac{\partial f}{\partial z} = ce^{ax+by+cz} \quad \frac{\partial^2 f}{\partial z^2} = c^2 e^{ax+by+cz}$$

$$\vec{\nabla} f = \left( a \vec{i} + b \vec{j} + c \vec{k} \right) e^{ax+by+cz}$$

$$\Delta f = (a^2 + b^2 + c^2) e^{ax+by+cz}$$

6.  $f(x, y, z) = (x^2 + y^2 + z^2)^n$

$$\frac{\partial f}{\partial x} = 2nx(x^2 + y^2 + z^2)^{n-1}$$

$$\frac{\partial f}{\partial y} = 2ny(x^2 + y^2 + z^2)^{n-1}$$

$$\frac{\partial f}{\partial z} = 2nz(x^2 + y^2 + z^2)^{n-1}$$

$$\vec{\nabla} f = 2n \left( x \vec{i} + y \vec{j} + z \vec{k} \right) (x^2 + y^2 + z^2)^{n-1}$$

$$\frac{\partial^2 f}{\partial x^2} = 2n(x^2 + y^2 + z^2)^{n-1} + 4n(n-1)x^2(x^2 + y^2 + z^2)^{n-2}$$

$$\frac{\partial^2 f}{\partial y^2} = 2n(x^2 + y^2 + z^2)^{n-1} + 4n(n-1)y^2(x^2 + y^2 + z^2)^{n-2}$$

$$\frac{\partial^2 f}{\partial z^2} = 2n(x^2 + y^2 + z^2)^{n-1} + 4n(n-1)z^2(x^2 + y^2 + z^2)^{n-2}$$

$$\Delta f = 6n(x^2 + y^2 + z^2)^{n-1} + 4n(n-1)(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{n-2}$$

$$= (6n + 4n(n-1))(x^2 + y^2 + z^2)^{n-1}$$

### Exercice 2

Calculer les dérivées directionnelles de la fonction  $f$  suivant la direction  $\vec{u}$  dans les cas suivantes :

1.  $f(x, y) = x^2y + 4y^2, \quad \vec{u} = 2\vec{i} + \vec{j}$
2.  $f(x, y) = \cos(2x - y), \quad \vec{u} = \overrightarrow{MN}, \text{ avec } M(\pi, 0) \quad N(2\pi, \pi)$
3.  $f(x, y, z) = x^2z - 2xy + y^2; \quad \vec{u} = \overrightarrow{MN}, \text{ avec } M(-2, -1) \quad N(2, -3)$
4.  $f(x, y, z) = y^2 + 2ye^{4xz}; \quad \vec{u} = \overrightarrow{MN}, \text{ avec } M(0, -2, 1) \quad N(-4, 4, 0)$
5.  $f(x, y, z) = x^2 + y^2 + z^2; \quad \vec{u} = \vec{i} + \vec{j} + \vec{k}$
6.  $f(x, y, z) = xyz \quad \vec{u} = \vec{i} + \vec{j} + \vec{k}$
7.  $f(x, y, z) = \frac{1}{r} \quad \vec{u} = \vec{r}$



**Solution 2** La dérivée directionnelle est définie par  $\frac{\partial f}{\partial u} = \overrightarrow{\nabla} f \cdot \vec{e}$  avec  $\vec{e} = \frac{\vec{u}}{u}$

1.  $\vec{u} = 2\vec{i} + \vec{j}, u = \sqrt{4+1} = \sqrt{5} \implies \vec{e} = \frac{2\vec{i} + \vec{j}}{\sqrt{5}}$   
 $\overrightarrow{\nabla} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} = 2xy \vec{i} + (x^2 + 8y) \vec{j} + 0 \cdot \vec{k}$   
 $\frac{\partial f}{\partial u} = \overrightarrow{\nabla} f \cdot \vec{e} = \frac{1}{\sqrt{5}} (2xy \vec{i} + (x^2 + 8y) \vec{j}) \cdot (2\vec{i} + \vec{j}) = \frac{1}{\sqrt{5}} (4xy + x^2 + 8y)$
2.  $\overrightarrow{\nabla} f = -2 \sin(2x - y) \vec{i} + \sin(2x - y) \vec{j},$   
 $\vec{u} = \overrightarrow{MN} = \pi \vec{i} + \pi \vec{j}, u = \pi\sqrt{2} \implies \vec{e} = \frac{1}{\sqrt{2}} (\vec{i} + \vec{j})$   
 $\frac{\partial f}{\partial u} = \frac{1}{\sqrt{2}} (-2 \sin(2x - y) + \sin(2x - y)) = -\frac{\sin(2x - y)}{\sqrt{2}}$
3.  $\overrightarrow{\nabla} f = (2xz - 2y) \vec{i} + (-2x + 2y) \vec{j} + x^2 \vec{k}$   
 $\vec{u} = \overrightarrow{MN} = 4\vec{i} - 2\vec{j}; u = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$   
 $\implies \vec{e} = \frac{1}{2\sqrt{5}} (4\vec{i} - 2\vec{j}) = \frac{1}{\sqrt{5}} (2\vec{i} - \vec{j})$   
 $\frac{\partial f}{\partial u} = \frac{1}{\sqrt{5}} (2(2xz - 2y) - (-2x + 2y)) = \frac{1}{\sqrt{5}} (2x - 6y + 4xz)$
4.  $\overrightarrow{\nabla} f = 8yze^{4xz} \vec{i} + (2y + 2e^{4xz}) \vec{j} + 8xye^{4xz} \vec{k}$   
 $\vec{u} = \overrightarrow{MN} = -4\vec{i} + 6\vec{j} - \vec{k}, u = \sqrt{16+36+1} = \sqrt{53}$   
 $\implies \vec{e} = \frac{1}{\sqrt{53}} (-4\vec{i} + 6\vec{j} - \vec{k})$   
 $\frac{\partial f}{\partial u} = \frac{1}{\sqrt{53}} (-32yze^{4xz} + 6(2y + 2e^{4xz}) - 8xye^{4xz})$   
 $= \frac{1}{\sqrt{53}} (12y + (12 - 8xy - 32yz) e^{4xz})$
5.  $\overrightarrow{\nabla} f = 2x \vec{i} + 2y \vec{j} + 2z \vec{k}$   
 $\vec{e} = \frac{1}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k})$   
 $\frac{\partial f}{\partial u} = \frac{2}{\sqrt{3}} (x + y + z)$
6.  $\overrightarrow{\nabla} f = yz \vec{i} + xz \vec{j} + xy \vec{k}$

$$\vec{e} = \frac{1}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k})$$

$$\frac{\partial f}{\partial u} = \frac{1}{\sqrt{3}} (yz + xz + xy)$$

$$7. f = \frac{1}{r} = (x^2 + y^2 + z^2)^{-1/2}$$

$$\frac{\partial f}{\partial x} = -\frac{1}{2} (2x) (x^2 + y^2 + z^2)^{-3/2} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{x}{r^3}$$

$$\text{de même : } \frac{\partial f}{\partial y} = -\frac{y}{r^3} \text{ et } \frac{\partial f}{\partial z} = -\frac{z}{r^3}$$

$$\vec{\nabla} f = -\frac{x}{r^3} \vec{i} - \frac{y}{r^3} \vec{j} - \frac{z}{r^3} \vec{k} = -\frac{\vec{r}}{r^3}$$

$$\vec{e} = \frac{\vec{u}}{u} = \frac{\vec{r}}{r}$$

$$\frac{\partial f}{\partial u} = \vec{\nabla} f \cdot \vec{e} = -\frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{r} = -\frac{r^2}{r^4} = -\frac{1}{r^2}$$

### Exercice 3

Déterminer la normale unitaire en un point  $M$ , des surfaces suivantes :

$$1. \text{ plan : } ax + by + cz + d = 0$$

$$2. \text{ Sphère } (O, R)$$

$$3. \text{ Ellipsoïde } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$4. f(x, y, z) = \ln(x + y + z)$$

$$5. f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

$$6. z = 2 - x^4 - y^4$$



**Solution 3** : La normale unitaire sur la surface définie par  $f(x, y, z) = 0$  est

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$$

$$1. f = ax + by + cz + d$$

$$\vec{\nabla} f = a \vec{i} + b \vec{j} + c \vec{k} \implies |\vec{\nabla} f| = \sqrt{a^2 + b^2 + c^2}$$

$$\vec{n} = \frac{a \vec{i} + b \vec{j} + c \vec{k}}{\sqrt{a^2 + b^2 + c^2}}$$

$$2. \text{ Equation de sphère } (O, R) : f(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$$

$$\vec{\nabla} f = 2x \vec{i} + 2y \vec{j} + 2z \vec{k} \implies |\vec{\nabla} f| = 2\sqrt{x^2 + y^2 + z^2} = 2R$$

$$\vec{n} = \frac{1}{R} (x \vec{i} + y \vec{j} + z \vec{k}) = \frac{\vec{r}}{R}$$

$$3. f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$\vec{\nabla} f = \frac{2x}{a^2} \vec{i} + \frac{2y}{b^2} \vec{j} + \frac{2z}{c^2} \vec{k}$$

$$|\vec{\nabla} f| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

$$\vec{n} = \frac{\frac{x}{a^2} \vec{i} + \frac{y}{b^2} \vec{j} + \frac{z}{c^2} \vec{k}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

$$4. \quad \vec{\nabla} f = \frac{\vec{i} + \vec{j} + \vec{k}}{x + y + z} \implies |\vec{\nabla} f| = \frac{\sqrt{3}}{x + y + z}$$

$$\vec{n} = \frac{1}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k})$$

$$5. \quad \frac{\partial f}{\partial x} = \frac{1}{2} (-2x) (1 - x^2 - y^2 - z^2)^{-1/2} = -\frac{x}{\sqrt{1 - x^2 - y^2 - z^2}}$$

$$\text{de même } \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1 - x^2 - y^2 - z^2}} \text{ et } \frac{\partial f}{\partial z} = -\frac{z}{\sqrt{1 - x^2 - y^2 - z^2}}$$

$$\vec{\nabla} f = -\frac{x \vec{i} + y \vec{j} + z \vec{k}}{\sqrt{1 - x^2 - y^2 - z^2}} \implies |\vec{\nabla} f| = \frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{1 - x^2 - y^2 - z^2}}$$

$$\vec{n} = -\frac{x \vec{i} + y \vec{j} + z \vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$6. \quad f = 2 - x^4 - y^4 - z$$

$$\vec{\nabla} f = -4x^3 \vec{i} - 4y^3 \vec{j} - \vec{k} \implies |\vec{\nabla} f| = \sqrt{16x^6 + 16y^6 + 1}$$

$$\vec{n} = \frac{-4x^3 \vec{i} - 4y^3 \vec{j} - \vec{k}}{\sqrt{16x^6 + 16y^6 + 1}}$$

#### Exercice 4

Soit  $\vec{H} = (x + 3y) \vec{i} + (y - 2z) \vec{j} + (x + az) \vec{k}$  et  $\varphi(x, y, z) = xy^3 \sin xz$

1. Déterminer la constante  $a$  pour que  $\text{div } \vec{H} = 0$ , et calculer dans ce cas  $\vec{\nabla} \times \vec{H}$ .
2. Calculer  $\vec{\nabla} \varphi$  et  $\Delta \varphi$
3. Calculer  $\vec{\nabla} \cdot (\varphi \vec{H})$



#### Solution 4 :

$$1. \quad \text{div } \vec{H} = \vec{\nabla} \cdot \vec{H} = \frac{\partial}{\partial x} (x + 3y) + \frac{\partial}{\partial y} (y - 2z) + \frac{\partial}{\partial z} (x + az) = 1 + 1 + a = 2 + a$$

$$\text{div } \vec{H} = 0 \iff a + 2 = 0 \implies a = -2 \text{ soit donc}$$

$$\vec{H} = (x + 3y) \vec{i} + (y - 2z) \vec{j} + (x - 2z) \vec{k} \quad (1)$$

$$\vec{\nabla} \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 3y & y - 2z & x - 2z \end{vmatrix} = 2 \vec{i} - \vec{j} - 3 \vec{k}$$

$$2. \varphi(x, y, z) = xy^3 \sin xz$$

$$\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial x} \vec{i} + \frac{\partial \varphi}{\partial y} \vec{j} + \frac{\partial \varphi}{\partial z} \vec{k}$$

$$= (y^3 \sin xz + xy^3 z \cos xz) \vec{i} + 3xy^2 \sin xz \vec{j} + x^2 y^3 \cos xz \vec{k}$$

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$$= (zy^3 \cos xz + y^3 z \cos xz - xy^3 z^2 \sin xz) + 6xy \sin xz - x^3 y^3 \sin xz$$

$$= 6xy \sin xz - x^3 y^3 \sin xz + 2y^3 z \cos xz - xy^3 z^2 \sin xz$$

$$\Delta \varphi = (6xy - x^3 y^3 - xy^3 z^2) \sin xz + 2y^3 z \cos xz$$

$$3. \vec{\nabla} \cdot (\varphi \vec{H}) = \varphi \vec{\nabla} \cdot \vec{H} + \vec{\nabla} \varphi \cdot \vec{H}$$

$$\text{On a pour } a = 2, \vec{\nabla} \cdot \vec{H} = 0 \implies \vec{\nabla} \cdot (\varphi \vec{H}) = \vec{\nabla} \varphi \cdot \vec{H}$$

$$= (x + 3y)(y^3 \sin xz + xy^3 z \cos xz) + (y - 2z) \times 3xy^2 \sin xz + (x - 2z)x^2 y^3 \cos xz$$

$$= 3y^4 \sin xz + x^3 y^3 \cos xz + 4xy^3 \sin xz - x^2 y^3 z \cos xz + 3xy^4 z \cos xz - 6xy^2 z \sin xz$$

$$\vec{\nabla} \cdot (\varphi \vec{H}) = (3y^4 + 4xy^3 - 6xy^2 z) \sin xz + (x^3 y^3 - x^2 y^3 z + 3xy^4 z) \cos xz$$

### Exercice 5

Soit  $\vec{H}$  un champ vectoriel défini dans l'espace rapporté au système  $(O, \vec{i}, \vec{j}, \vec{k})$

$$1. \text{ Calculer les produits vectoriels : } \vec{i} \times \vec{j}, \vec{i} \times \vec{k} \text{ et } \vec{j} \times \vec{k}$$

$$2. \text{ Montrer que}$$

$$\vec{\nabla} \times \vec{H} = \vec{i} \times \frac{\partial \vec{H}}{\partial x} + \vec{j} \times \frac{\partial \vec{H}}{\partial y} + \vec{k} \times \frac{\partial \vec{H}}{\partial z}$$

$$3. \text{ Application : } \vec{H} = (x^2 + y^2) \vec{i} + (y^2 + z^2) \vec{j} + (x^2 + z^2) \vec{k}$$



### Solution 5 :

$$1. (O, \vec{i}, \vec{j}, \vec{k}) \text{ est un système direct, alors}$$

$$\vec{i} \times \vec{j} = \vec{k}, \vec{i} \times \vec{k} = -\vec{j} \text{ et } \vec{j} \times \vec{k} = \vec{i}$$

de plus

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$$

$$2. \text{ Soit } \vec{H} = P \vec{i} + Q \vec{j} + R \vec{k}$$

$$\vec{i} \times \frac{\partial \vec{H}}{\partial x} = \vec{i} \times \left( \frac{\partial P}{\partial x} \vec{i} + \frac{\partial Q}{\partial x} \vec{j} + \frac{\partial R}{\partial x} \vec{k} \right)$$

$$= \frac{\partial P}{\partial x} \underbrace{\vec{i} \times \vec{i}}_{=\vec{0}} + \frac{\partial Q}{\partial x} \underbrace{\vec{i} \times \vec{j}}_{=\vec{k}} + \frac{\partial R}{\partial x} \underbrace{\vec{i} \times \vec{k}}_{=-\vec{j}} = \frac{\partial Q}{\partial x} \vec{k} - \frac{\partial R}{\partial x} \vec{j}$$

$$\vec{j} \times \frac{\partial \vec{H}}{\partial y} = \vec{j} \times \left( \frac{\partial P}{\partial y} \vec{i} + \frac{\partial Q}{\partial y} \vec{j} + \frac{\partial R}{\partial y} \vec{k} \right) = -\frac{\partial P}{\partial y} \vec{k} + \frac{\partial R}{\partial y} \vec{i}$$

$$\begin{aligned}\vec{k} \times \frac{\partial \vec{H}}{\partial z} &= \vec{k} \times \left( \frac{\partial P}{\partial z} \vec{i} + \frac{\partial Q}{\partial z} \vec{j} + \frac{\partial R}{\partial z} \vec{k} \right) = \frac{\partial P}{\partial z} \vec{j} - \frac{\partial Q}{\partial z} \vec{i} \\ \vec{i} \times \frac{\partial \vec{H}}{\partial x} + \vec{j} \times \frac{\partial \vec{H}}{\partial y} + \vec{k} \times \frac{\partial \vec{H}}{\partial z} &= \frac{\partial Q}{\partial x} \vec{k} - \frac{\partial R}{\partial x} \vec{j} - \frac{\partial P}{\partial y} \vec{k} + \frac{\partial R}{\partial y} \vec{i} + \frac{\partial P}{\partial z} \vec{j} - \frac{\partial Q}{\partial z} \vec{i} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} = \vec{\nabla} \times \vec{H}\end{aligned}$$

**3.** Application :

$$\begin{aligned}\vec{\nabla} \times \vec{H} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & y^2 + z^2 & x^2 + z^2 \end{vmatrix} = -2z \vec{i} - 2x \vec{j} - 2y \vec{k} \\ \vec{i} \times \frac{\partial \vec{H}}{\partial x} &= \vec{i} \times (2x \vec{i} + 2x \vec{k}) = -2x \vec{j} \\ \vec{j} \times \frac{\partial \vec{H}}{\partial y} &= \vec{j} \times (2y \vec{i} + 2y \vec{j}) = -2y \vec{k} \\ \vec{k} \times \frac{\partial \vec{H}}{\partial z} &= \vec{k} \times (2z \vec{j} + 2y \vec{k}) = -2z \vec{i} \\ \vec{i} \times \frac{\partial \vec{H}}{\partial x} + \vec{j} \times \frac{\partial \vec{H}}{\partial y} + \vec{k} \times \frac{\partial \vec{H}}{\partial z} &= -2z \vec{i} - 2x \vec{j} - 2y \vec{k} = \vec{\nabla} \times \vec{H}\end{aligned}$$

○ Pour calculer rapidement le rotationnel de  $\vec{H}$  on peut utiliser la notation suivante :

$$\begin{aligned}\vec{\nabla} \times \vec{H} &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \\ \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \searrow \\ \frac{\partial}{\partial z} \nearrow \end{pmatrix} \begin{pmatrix} P \\ \nearrow Q \\ \searrow R \end{pmatrix} &= \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \quad \begin{pmatrix} \frac{\partial}{\partial x} \searrow \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \nearrow \end{pmatrix} \begin{pmatrix} \nearrow P \\ Q \\ \searrow R \end{pmatrix} = - \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \\ \text{et } \begin{pmatrix} \frac{\partial}{\partial x} \searrow \\ \frac{\partial}{\partial y} \nearrow \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \nearrow P \\ \searrow Q \\ R \end{pmatrix} &= \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\end{aligned}$$

### Exercice 6

Soit  $\vec{v}$  ( $M$ ) le champ de vitesses des points d'un solide qui tourne, autour d'un axe fixe ( $D$ ), avec la vitesse de rotation  $\vec{\omega}$  constante. Dans ce cas on a  $\vec{v} = \vec{\omega} \times \vec{r}$ , à condition que l'axe de rotation passe par l'origine de coordonnées de  $\vec{r}$  des points  $M$  de solide.

Si on considère le cas où  $\vec{\omega} = \omega \vec{k}$  et  $\vec{r} = x \vec{i} + y \vec{j}$

**1.** Calculer les composantes de  $\vec{v}$

2. Calculer  $\text{div } \vec{v}$  et montrer que  $\vec{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{v}$



**Solution 6 :**

$$1. \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = -\omega y \vec{i} + \omega x \vec{j} = v_x \vec{i} + v_y \vec{j}$$

$$2. \text{div } \vec{v} = \vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\begin{aligned} \vec{\nabla} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = (0-0) \vec{i} - (0-0) \vec{j} + (\omega + \omega) \vec{k} \\ &= 2\omega \vec{k} = 2\vec{\omega} \iff \vec{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{v} \end{aligned}$$

### Exercice 7

On considère les champs scalaires :

$$\begin{aligned} f(x, y, z, t) &= Ae^{j(\omega t - ax - by - cz)} \\ g(x, y, z, t) &= \sin n\pi(x + y + z) \cos n\pi\omega t \end{aligned}$$

où  $A, \omega, a, b, c$  et  $n$  sont des constantes données,  $x, y, z, t$  sont des variables réelles.

On pose  $k = \sqrt{a^2 + b^2 + c^2}$  et  $v = \frac{\omega}{k}$ . Montrer que :

$$1. \Delta f - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0$$

$$2. \Delta g - \frac{3}{\omega^2} \frac{\partial^2 g}{\partial t^2} = 0$$



**Solution 7 :**

$$\begin{aligned} 1. \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (A \exp(j(\omega t - ax - by - cz))) \\ &= -jaA \exp(j(\omega t - ax - by - cz)) = -jaf \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} (-jaf) = -ja \frac{\partial f}{\partial x} = (-ja)^2 f = -a^2 f \end{aligned}$$

de même on trouve :

$$\frac{\partial^2 f}{\partial y^2} = -b^2 f \quad \frac{\partial^2 f}{\partial z^2} = -c^2 f$$

$$\text{donc } \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -(a^2 + b^2 + c^2) f = -k^2 f$$

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 f$$



On a  $f = -\frac{1}{k^2}\Delta f = -\frac{1}{\omega^2}\frac{\partial^2 f}{\partial t^2} \implies \Delta f = \frac{k^2}{\omega^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$   
d'où finalement :

$$\Delta f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

2.  $g(x, y, z, t) = \sin n\pi(x + y + z) \cos n\pi\omega t$

$$\frac{\partial^2}{\partial t^2} (\sin n\pi(x + y + z) \cos n\pi\omega t) = -\pi^2\omega^2 n^2 \sin n\pi(x + y + z) \cos n\pi\omega t$$

$$= -\pi^2 n^2 \omega^2 g$$

$$\frac{\partial^2}{\partial x^2} (\sin n\pi(x + y + z) \cos n\pi\omega t) = -\pi^2 n^2 \sin n\pi(x + y + z) \cos n\pi\omega t$$

$$= -\pi^2 n^2 g$$

alors :  $\Delta g = -3\pi^2 n^2 g$

Remarquons que :  $-\pi^2 n^2 g = \frac{\Delta g}{3} = \frac{1}{\omega^2} \frac{\partial^2 g}{\partial t^2}$  donc :

$$\Delta g - \frac{3}{\omega^2} \frac{\partial^2 g}{\partial t^2} = 0$$

### Exercice 8

Une charge électrique  $q$  est soumise, du côté d'une charge fixe  $Q$  à la force de Coulombe :

$$\vec{F} = \frac{qQ}{4\pi\epsilon} \frac{\vec{r}}{r^3}$$

1. Calculer :

(a)  $\text{div } \vec{F}$

(b)  $\vec{\nabla} \times \vec{F}$

2. Montrer que  $\vec{F}$  dérive d'un potentiel scalaire  $\varphi = \varphi(x, y, z)$  et déterminer l'expression de  $\varphi$  si  $\varphi(\infty) = 0$



### Solution 8 :

1.  $\vec{F} = \frac{qQ}{4\pi\epsilon} \frac{\vec{r}}{r^3} = \frac{qQ}{4\pi\epsilon} \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}$

(a)  $\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

$$\frac{\partial F_x}{\partial x} = \frac{qQ}{4\pi\epsilon} \frac{\partial}{\partial x} \left( \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) = \frac{Qq}{4\pi\epsilon} \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \frac{Qq}{4\pi\epsilon} \frac{-3x^2 + x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} = \frac{Qq}{4\pi\epsilon} \frac{-3x^2 + r^2}{r^5}$$

$$\frac{\partial F_y}{\partial y} = \frac{qQ}{4\pi\epsilon} \frac{\partial}{\partial y} \left( \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \right)$$

$$\begin{aligned}
&= \frac{Qq}{4\pi\epsilon} \frac{x^2 - 2y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} = \frac{Qq}{4\pi\epsilon} \frac{-3y^2 + r^2}{r^5} \\
\frac{\partial F_z}{\partial z} &= \frac{qQ}{4\pi\epsilon} \frac{\partial}{\partial z} \left( \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) \\
&= \frac{Qq}{4\pi\epsilon} \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}} = \frac{Qq}{4\pi\epsilon} \frac{-3z^2 + r^2}{r^5} \\
\operatorname{div} \vec{F} &= \frac{Qq}{4\pi\epsilon} \left( \frac{-3x^2 + r^2}{r^5} + \frac{-3y^2 + r^2}{r^5} + \frac{-3z^2 + r^2}{r^5} \right) \\
&= \frac{Qq}{4\pi\epsilon} \frac{-3(x^2 + y^2 + z^2) + r^2}{r^5} = 0
\end{aligned}$$

$$(b) \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y} \left( \frac{qQ}{4\pi\epsilon} \frac{x}{(\sqrt{x^2 + y^2 + z^2})^3} \right) = -\frac{3Qq}{4\pi\epsilon} \frac{xy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x} \left( \frac{qQ}{4\pi\epsilon} \frac{y}{(\sqrt{x^2 + y^2 + z^2})^3} \right) = -\frac{3Qq}{4\pi\epsilon} \frac{xy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

$$\implies \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

$$\text{de même on trouve } \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x} \text{ et } \frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z} \implies \vec{\nabla} \times \vec{F} = \vec{0}$$

2. Puisque  $\vec{\nabla} \times \vec{F} = \vec{0}$  donc  $\vec{F}$  dérive d'un potentiel scalaire, c'est-à-dire il existe un champ scalaire  $\varphi = \varphi(x, y, z)$  tel que  $\vec{F} = \vec{\nabla} \varphi$

$$(a) \quad F_x = \frac{qQ}{4\pi\epsilon} \frac{x}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial \varphi}{\partial x}$$

$$\implies \varphi = \int \frac{qQ}{4\pi\epsilon} \frac{xdx}{(\sqrt{x^2 + y^2 + z^2})^3} = -\frac{qQ}{4\pi\epsilon} \frac{1}{\sqrt{x^2 + y^2 + z^2}} + f(y, z)$$

$$F_y = \frac{qQ}{4\pi\epsilon} \frac{y}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial \varphi}{\partial y}$$

$$= \frac{\partial}{\partial y} \left( -\frac{qQ}{4\pi\epsilon} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) + \frac{\partial f}{\partial y} = \frac{Qq}{4\pi\epsilon} \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{\partial f}{\partial y}$$

$$\implies \frac{\partial f}{\partial y} = 0 \implies f = f(z)$$

$$\varphi = -\frac{qQ}{4\pi\epsilon} \frac{1}{\sqrt{x^2 + y^2 + z^2}} + f(z)$$

$$F_z = \frac{qQ}{4\pi\epsilon} \frac{z}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{\partial \varphi}{\partial z}$$

$$= \frac{\partial}{\partial z} \left( -\frac{qQ}{4\pi\epsilon} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) + \frac{\partial f}{\partial z} = \frac{Qq}{4\pi\epsilon} \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{\partial f}{\partial z}$$

$$\implies \frac{\partial f}{\partial z} = 0 \implies f = \text{Const} = C$$

$$\varphi = -\frac{qQ}{4\pi\epsilon} \frac{1}{\sqrt{x^2 + y^2 + z^2}} + C = -\frac{qQ}{4\pi\epsilon r} + C$$

○ On peut calculer  $\varphi$  directement en fonction de  $r$ ; en effet :

$$\text{Soit } \vec{dr} = dx \vec{i} + dy \vec{j} + dz$$

$$\vec{F} = \vec{\nabla} \varphi \implies \vec{F} \cdot \vec{dr} = \vec{\nabla} \varphi \cdot \vec{dr} = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = d\varphi$$

$$\text{donc } d\varphi = \vec{F} \cdot \vec{dr} = \frac{qQ}{4\pi\epsilon} \frac{\vec{r}}{r^3} \cdot \vec{dr} = \frac{qQ}{4\pi\epsilon} \frac{r dr}{r^3} = \frac{qQ}{4\pi\epsilon} \frac{dr}{r^2}$$

$$\varphi = \int \frac{qQ}{4\pi\epsilon} \frac{dr}{r^2} = -\frac{1}{4\pi\epsilon} \frac{qQ}{r} + C$$

### Exercice 9

On considère le champ vectoriel :

$$\vec{F}(M) = \frac{\vec{r}}{r^2 + 1}$$

1. Calculer  $\vec{\nabla} \cdot \vec{F}$  et  $\vec{\nabla} \times \vec{F}$
2.  $\vec{F}$  est-il un champ conservatif? Justifier la réponse.
3. Déterminer le potentiel scalaire  $\varphi = \varphi(x, y, z)$  de  $\vec{F}$  tel que  $\varphi(0) = 0$ .



$$\text{Solution 9 : } \vec{F}(M) = \frac{\vec{r}}{r^2 + 1} = \frac{x \vec{i} + y \vec{j} + z \vec{k}}{x^2 + y^2 + z^2 + 1} = P \vec{i} + Q \vec{j} + R \vec{k}$$

$$1. \vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2 + z^2 + 1} \right) = \frac{-x^2 + y^2 + z^2 + 1}{(x^2 + y^2 + z^2 + 1)^2} = \frac{r^2 + 1 - 2x^2}{(r^2 + 1)^2}$$

$$\text{de même } \frac{\partial Q}{\partial y} = \frac{r^2 + 1 - 2y^2}{(r^2 + 1)^2}$$

$$\text{et } \frac{\partial R}{\partial z} = \frac{r^2 + 1 - 2z^2}{(r^2 + 1)^2}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{r^2 + 1 - 2x^2}{(r^2 + 1)^2} + \frac{r^2 + 1 - 2y^2}{(r^2 + 1)^2} + \frac{r^2 + 1 - 2z^2}{(r^2 + 1)^2} = \frac{r^2 + 3}{(r^2 + 1)^2}$$

- On peut utiliser aussi la formule :  $\text{div}(u \vec{V}) = \vec{\text{grad}} u \cdot \vec{V} + u \text{div} \vec{V}$

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^2 + 1} = \vec{\nabla} \left( \frac{1}{r^2 + 1} \right) \cdot \vec{r} + \frac{\vec{\nabla} \cdot \vec{r}}{r^2 + 1}$$

$$= \frac{-2x \vec{i} - 2y \vec{j} - 2z \vec{k}}{(r^2 + 1)^2} \cdot \vec{r} + \frac{3}{r^2 + 1}$$

$$= \frac{-2r^2}{(r^2 + 1)^2} + \frac{3}{r^2 + 1} = \frac{r^2 + 3}{(r^2 + 1)^2}$$

$$\vec{\nabla} \times \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y} \left( \frac{z}{x^2 + y^2 + z^2 + 1} \right) = \frac{-2yz}{(x^2 + y^2 + z^2 + 1)^2} = \frac{\partial Q}{\partial z}$$

de même on trouve

$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = \frac{-2xz}{(x^2 + y^2 + z^2 + 1)^2}$$

$$\text{et } \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{-2xy}{(x^2 + y^2 + z^2 + 1)^2}$$

$$\text{Donc } \vec{\nabla} \times \vec{F} = \vec{0}$$

2.  $\vec{\nabla} \times \vec{F} = \vec{0}$  donc  $\vec{F}$  est un champ conservatif

3.  $\exists \varphi = \varphi(x, y, z) / \vec{F} = \vec{\nabla} \varphi$

$$\implies \vec{F} \cdot d\vec{r} = \vec{\nabla} \varphi \cdot d\vec{r} = d\varphi$$

$$\varphi = \int d\varphi = \int \vec{F} \cdot d\vec{r} = \int \frac{\vec{r} \cdot d\vec{r}}{r^2 + 1}$$

$$= \int \frac{r dr}{r^2 + 1} = \frac{1}{2} \ln(r^2 + 1) + C$$

$$\varphi(0) = \frac{1}{2} \ln(0 + 1) + C = 0 \implies C = 0$$

$$\varphi(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2 + 1}$$

### Exercice 10

On considère le champ vectoriel

$$\vec{H} = \vec{r} \exp(-r^2)$$

1. Calculer  $\vec{\nabla} \cdot \vec{H}$  et  $\vec{\nabla} \times \vec{H}$ .

2. Montrer que  $\vec{H}$  est un champ de gradient.

3. Déterminer le champ scalaire  $f(x, y, z)$  tel que  $\vec{H} = \vec{\nabla} f$  et  $f(\infty, \infty, \infty) = 0$



$$\text{Solution 10 } \vec{H} = \vec{r} \exp(-r^2) = P \vec{i} + Q \vec{j} + R \vec{k}$$

$$1. \vec{\nabla} \cdot \vec{H} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$P = x e^{-x^2 - y^2 - z^2} \implies \frac{\partial P}{\partial x} = e^{-x^2 - y^2 - z^2} - 2x^2 e^{-x^2 - y^2 - z^2} = (1 - 2x^2) e^{-x^2 - y^2 - z^2}$$

$$Q = y e^{-x^2 - y^2 - z^2} \implies \frac{\partial Q}{\partial y} = e^{-x^2 - y^2 - z^2} - 2y^2 e^{-x^2 - y^2 - z^2} = (1 - 2y^2) e^{-x^2 - y^2 - z^2}$$

$$R = z e^{-x^2 - y^2 - z^2} \implies \frac{\partial R}{\partial z} = (1 - 2z^2) e^{-x^2 - y^2 - z^2} \text{ alors}$$

$$\vec{\nabla} \cdot \vec{H} = (3 - 2r^2) e^{-r^2}$$

$$\vec{\nabla} \times \vec{H} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

$$\frac{\partial R}{\partial y} = -2yze^{-x^2-y^2-z^2} \quad \frac{\partial Q}{\partial z} = -2yze^{-x^2-y^2-z^2} \implies \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0$$

de même on trouve :  $\left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$  par suite

$$\vec{\nabla} \times \vec{H} = \vec{0}$$

2.  $\vec{\nabla} \times \vec{H} = \vec{0} \implies \vec{H}$  est un champ de gradient.

3.  $\vec{H}$  est un champ de gradient, c-à-d il existe une fonction  $f(x, y, z)$  ;  $\vec{H} = \vec{\nabla} f$

C'est-à-dire :  $\vec{H} \cdot d\vec{r} = df$

$$df = \vec{H} \cdot d\vec{r} = \vec{r} \exp(-r^2) \cdot d\vec{r} = r \exp(-r^2) dr$$

$$f = \int r \exp(-r^2) dr = -\frac{1}{2} e^{-r^2} + C$$

$f(\infty) = C = 0$  d'où finalement :

$$f(x, y, z) = -\frac{1}{2} e^{-r^2} = -\frac{1}{2} e^{-x^2-y^2-z^2}$$

### Exercice 11

On considère le champ vectoriel :

$$\vec{W} = (x^\alpha + xy^2 + xz^2) \vec{i} + (x^2y + y^\beta + yz^2) \vec{j} + (x^2z + y^2z + z^\gamma) \vec{k}$$

1. Calculer  $\overrightarrow{rotW}$ .

2. Pour quelles valeurs de  $\alpha, \beta$ , et  $\gamma$  le champ  $\vec{W}$  est un champ de gradient ?

3. Soit  $\alpha = \beta = \gamma = 3$ .

Exprimer  $\vec{W}$  à l'aide du rayon vecteur  $\vec{r} = \overrightarrow{OM}$ . et  $r = \|\vec{r}\|$ .

4. Déterminer dans les conditions 3) le champ scalaire  $U$  ;  $\vec{W} = \overrightarrow{grad U}$ .



### Solution 11 :

$$1. \vec{\nabla} \times \vec{W} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^\alpha + xy^2 + xz^2 & x^2y + y^\beta + yz^2 & x^2z + y^2z + z^\gamma \end{vmatrix}$$

$$= (2yz - 2yz) \vec{i} - (2xz - 2xz) \vec{j} + (2xy - 2xy) \vec{k} = \vec{0}$$

2. On a  $\vec{\nabla} \times \vec{W} = \vec{0}$  indépendamment des valeurs de  $\alpha, \beta$ , et  $\gamma$  alors  $\vec{W}$  est un champ de gradient  $\forall \alpha, \beta, \gamma \in \mathbb{R}$ .

3.  $\alpha = \beta = \gamma = 3 \implies$

$$\vec{W} = (x^3 + xy^2 + xz^2) \vec{i} + (x^2y + y^3 + yz^2) \vec{j} + (x^2z + y^2z + z^3) \vec{k}$$

$$= x(x^2 + y^2 + z^2) \vec{i} + y(x^2 + y^2 + z^2) \vec{j} + z(x^2 + y^2 + z^2) \vec{k}$$

$$= (x^2 + y^2 + z^2) (x \vec{i} + y \vec{j} + z \vec{k}) = r^2 \vec{r}$$

$$4. \vec{W} = \overrightarrow{\text{grad } U} = \frac{\partial U}{\partial x} \vec{i} + \frac{\partial U}{\partial y} \vec{j} + \frac{\partial U}{\partial z} \vec{k}$$

$$\frac{\partial U}{\partial x} = x^3 + xy^2 + xz^2$$

$$\implies U(x, y, z) = \int (x^3 + xy^2 + xz^2) dx = \frac{1}{4}x^2(x^2 + 2y^2 + 2z^2) + f(y, z)$$

$$\frac{\partial U}{\partial y} = x^2y + \frac{\partial f}{\partial y} = x^2y + y^3 + z^2y \implies \frac{\partial f}{\partial y} = y^3 + z^2y$$

$$\implies f(y, z) = \int (y^3 + z^2y) dy = \frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + g(z)$$

$$U(x, y, z) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}x^2z^2 + \frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + g(z)$$

$$\frac{\partial U}{\partial z} = x^2z + y^2z + g' = x^2z + y^2z + z^3$$

$$\implies g' = z^3 \implies g(z) = \frac{z^4}{4} + C$$

$$U(x, y, z) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}x^2z^2 + \frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + \frac{1}{4}z^4 + C$$

$$= \frac{1}{4}(x^4 + 2x^2y^2 + 2x^2z^2 + y^4 + 2y^2z^2 + z^4) + C = \frac{r^4}{4} + C$$

$$\bigcirc \vec{W} = r^2 \vec{r} \implies dU = \vec{W} \cdot \vec{dr} = r^3 dr \implies U = \frac{r^4}{4} + C$$

### Exercice 12

On considère le champ de vecteurs :

$$\vec{q} = (6xy + \beta z^3) \vec{i} + (3x^2 - z) \vec{j} + (3xz^2 - y + z \sin z) \vec{k}$$

1. Calculer  $\text{div } \vec{q}$  et  $\overrightarrow{\text{rot } \vec{q}}$

2. Pour quelle valeur de  $\beta$ ,  $\vec{q}$  est un champ de gradient ?

3. Avec la valeur de  $\beta$  ainsi trouvée, déterminer le champ  $F$  tel que  $\vec{q} = \overrightarrow{\nabla} F$  et  $F(0, 0, 0) = 3$



### Solution 12 :

1.  $\overrightarrow{\nabla} \cdot \vec{q} = \text{div } \vec{q} = \frac{\partial}{\partial x} (6xy + \beta z^3) + \frac{\partial}{\partial y} (3x^2 - z) + \frac{\partial}{\partial z} (3xz^2 - y + z \sin z)$

$$= 6y + 6xz + \sin z + z \cos z$$

$$\nabla \times \vec{q} = (3\beta z^2 - 3z^2) \vec{j}$$

2.  $\vec{q}$  est un champ de gradient c.à.d. il existe  $F(M)$  tel que

$$\vec{q} = \overrightarrow{\text{grad } F} \iff \overrightarrow{\nabla} \times \vec{q} = \vec{0}$$

$$\nabla \times \vec{q} = (3\beta z^2 - 3z^2) \vec{j} \Rightarrow 3\beta z^2 - 3z^2 = 0 \Rightarrow \beta = 1$$

Pour  $\beta = 1$  :

$$\vec{q} = (6xy + z^3) \vec{i} + (3x^2 - z) \vec{j} + (3xz^2 - y + z \sin z) \vec{k}$$

$$\begin{aligned}
 3. \quad \vec{q} &= \vec{\nabla} F = \frac{\partial F}{\partial x} \vec{i} + \frac{\partial F}{\partial y} \vec{j} + \frac{\partial F}{\partial z} \vec{k} \\
 &\Rightarrow \begin{cases} F'_x = 6xy + z^3 \\ F'_y = 3x^2 - z \\ F'_z = 3xz^2 - y + z \sin z \end{cases} \\
 F'_x = 6xy + z^3 &\Rightarrow F(x, y, z) = \int (6xy + z^3) dx = 3x^2y + z^3x + f(y, z) \\
 F'_y = 3x^2 - z = 3x^2 + \frac{\partial f}{\partial y} &\Rightarrow \frac{\partial f}{\partial y} = -z \Rightarrow f(y, z) = -yz + g(z) \\
 \text{donc } F &= 3x^2y + z^3x - yz + g(z) \\
 F'_z = 3xz^2 - y + z \sin z = 3z^2x - y + g' &\Rightarrow g' = z \sin z \\
 \Rightarrow g &= \int z \sin z dz = \sin z - z \cos z + C \\
 \text{et } F &= 3x^2y + z^3x - yz + \sin z - z \cos z + C \\
 F(0, 0, 0) &= C = 3 \\
 F &= 3x^2y + z^3x - yz + \sin z - z \cos z + 3
 \end{aligned}$$

### Exercice 13

Soit  $u(M)$  un champ scalaire défini et continu dans un domaine  $D$ , et  $\vec{V} = \vec{V}(M)$  un champ vectoriel.

1. Montrer que, si  $f(u)$  est une fonction dérivable de  $u$  alors  $\vec{\nabla} f = \frac{df}{du} \vec{\nabla} u$
2. Démontrer la formule :  $\operatorname{div}(u\vec{V}) = \overrightarrow{\operatorname{grad}} u \cdot \vec{V} + u \cdot \operatorname{div} \vec{V}$ .
3. Démontrer que  $\forall n \in \mathbb{Q}; \vec{\nabla} r^n = nr^{n-2} \vec{r}$
4. Dédurre :  $\vec{\nabla} r, \vec{\nabla} \left(\frac{1}{r}\right), \vec{\nabla} \left(\frac{1}{r^2}\right), \Delta \left(\frac{1}{r}\right)$
5. Soit  $\vec{H} = \vec{r} f(r)$ . Déterminer  $f(r)$  telle que  $\operatorname{div} \vec{H} = 0$  et  $f(1) = 1$



### Solution 13 :

1. 
$$\begin{aligned}
 \vec{\nabla} f &= \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} = \frac{df}{du} \frac{\partial u}{\partial x} \vec{i} + \frac{df}{du} \frac{\partial u}{\partial y} \vec{j} + \frac{df}{du} \frac{\partial u}{\partial z} \vec{k} \\
 &= \frac{df}{du} \left( \frac{\partial u}{\partial x} \vec{i} + \frac{\partial u}{\partial y} \vec{j} + \frac{\partial u}{\partial z} \vec{k} \right) = \frac{df}{du} \vec{\nabla} u
 \end{aligned}$$
2. Soit  $\vec{V} = P\vec{i} + Q\vec{j} + R\vec{k} \Rightarrow u\vec{V} = uP\vec{i} + uQ\vec{j} + uR\vec{k}$ 

$$\begin{aligned}
 \vec{\nabla} \cdot (u\vec{V}) &= \frac{\partial(uP)}{\partial x} + \frac{\partial(uQ)}{\partial y} + \frac{\partial(uR)}{\partial z} = u \frac{\partial P}{\partial x} + \frac{\partial u}{\partial x} P + u \frac{\partial Q}{\partial y} + \frac{\partial u}{\partial y} Q + u \frac{\partial R}{\partial z} + \frac{\partial u}{\partial z} R \\
 &= u \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + u \frac{\partial R}{\partial z} \right) + \frac{\partial u}{\partial x} P + \frac{\partial u}{\partial y} Q + \frac{\partial u}{\partial z} R \\
 &= u \vec{\nabla} \cdot \vec{V} + \vec{\nabla} u \cdot \vec{V}
 \end{aligned}$$
3. On utilise la relation  $\vec{\nabla} f = \frac{df}{du} \vec{\nabla} u$  avec  $f = r^n, u = r$ 

$$\begin{aligned}
 \vec{\nabla} r^n &= \frac{dr^n}{dr} \vec{\nabla} r = nr^{n-1} \vec{\nabla} r \\
 \vec{\nabla} r &= \frac{\partial r}{\partial x} \vec{i} + \frac{\partial r}{\partial y} \vec{j} + \frac{\partial r}{\partial z} \vec{k}
 \end{aligned}$$

$$r = \sqrt{x^2 + y^2 + z^2} \implies \frac{\partial r}{\partial x} = \frac{1}{2} (2x) (x^2 + y^2 + z^2)^{-1/2} = \frac{x}{r}$$

$$\text{de même } \frac{\partial r}{\partial y} = \frac{y}{r}, \text{ et } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\vec{\nabla} r = \frac{x}{r} \vec{i} + \frac{y}{r} \vec{j} + \frac{z}{r} \vec{k} = \frac{\vec{r}}{r}$$

$$\vec{\nabla} r^n = nr^{n-1} \frac{\vec{r}}{r} = nr^{n-2} \vec{r}$$

$$4. \quad (a) \quad n = 1 \implies \vec{\nabla} r = r^{1-2} \vec{\nabla} r = \frac{\vec{r}}{r}$$

$$(b) \quad n = -1 \implies \vec{\nabla} \left( \frac{1}{r} \right) = -1r^{-1-2} \vec{r} = -\frac{\vec{r}}{r^3}$$

$$(c) \quad n = -2 \implies \vec{\nabla} \left( \frac{1}{r^2} \right) = -2r^{-2-2} \vec{r} = -2 \frac{\vec{r}}{r^4}$$

$$(d) \quad \Delta \left( \frac{1}{r} \right) = \vec{\nabla} \cdot \vec{\nabla} \left( \frac{1}{r} \right) = \vec{\nabla} \cdot \left( -\frac{\vec{r}}{r^3} \right)$$

$$= -\vec{\nabla} \left( \frac{1}{r^3} \right) \cdot \vec{r} - \frac{1}{r^3} \vec{\nabla} \cdot \vec{r}$$

$$= - \left( -3 \frac{1}{r^5} \vec{r} \right) \cdot \vec{r} - \frac{3}{r^3} = 3 \frac{r^2}{r^5} - \frac{3}{r^3} = 0$$

5. D'après la formule de question 2) et la relation de question 1)

$$\vec{\nabla} \cdot \vec{H} = \vec{\nabla} \cdot (\vec{r} f(r)) = f \vec{\nabla} \cdot \vec{r} + \vec{\nabla} f \cdot \vec{r}$$

$$= f \times (3) + \left( \frac{df}{dr} \vec{\nabla} r \right) \cdot \vec{r}$$

$$= 3f + \frac{df}{dr} \frac{\vec{r}}{r} \cdot \vec{r} = 3f + r \frac{df}{dr}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \implies 3f + r \frac{df}{dr} = 0 \implies \frac{df}{f} = -3 \frac{dr}{r} \implies f(r) = \frac{K}{r^3}$$

$$f(1) = 1 \implies K = 1 \implies f(r) = \frac{1}{r^3}$$

#### Exercice 14

Soit  $f(r)$  une fonction deux fois dérivable par rapport à  $r$ . On désigne par  $f'(r)$  et  $f''(r)$  les dérivées première et seconde de  $f(r)$ .

$$1. \quad \text{Démontrer que } \vec{\text{grad}} f = f'(r) \frac{\vec{r}}{r}$$

$$2. \quad \text{Calculer } \Delta f \text{ et exprimer } \Delta f \text{ en fonction de } f' \text{ et } f''$$

$$3. \quad \text{On pose } g = f', \text{ exprimer } \Delta f \text{ en fonction de } g \text{ et } g', \text{ puis résoudre l'équation } \Delta f = 0.$$



#### Solution 14 :

$$1. \quad \vec{\nabla} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} = \frac{df}{dr} \frac{\partial r}{\partial x} \vec{i} + \frac{df}{dr} \frac{\partial r}{\partial y} \vec{j} + \frac{df}{dr} \frac{\partial r}{\partial z} \vec{k}$$

$$= \frac{df}{dr} \left( \frac{\partial r}{\partial x} \vec{i} + \frac{\partial r}{\partial y} \vec{j} + \frac{\partial r}{\partial z} \vec{k} \right) = \frac{df}{dr} \vec{\nabla} r$$



$$r = \sqrt{x^2 + y^2 + z^2} \implies \frac{\partial r}{\partial x} = \frac{1}{2} (2x) (x^2 + y^2 + z^2)^{-1/2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

$$\text{de même } \frac{\partial r}{\partial y} = \frac{y}{r}, \text{ et } \frac{\partial r}{\partial z} = \frac{z}{r} \implies \vec{\nabla} r = \frac{x}{r} \vec{i} + \frac{y}{r} \vec{j} + \frac{z}{r} \vec{k} = \frac{\vec{r}}{r}$$

$$\text{d'où : } \vec{\nabla} f = \frac{df}{dr} \frac{\vec{r}}{r}$$

$$2. \Delta f = \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla} \cdot \left( \frac{df}{dr} \frac{\vec{r}}{r} \right) = \vec{\nabla} \left( \frac{f'}{r} \right) \cdot \vec{r} + \frac{f'}{r} \vec{\nabla} \cdot \vec{r}$$

$$= \frac{1}{r} \vec{\nabla} f' \cdot \vec{r} + f' \vec{\nabla} \left( \frac{1}{r} \right) \cdot \vec{r} + \frac{f'}{r} \vec{\nabla} \cdot \vec{r}$$

$$\vec{\nabla} f' = \frac{df'}{dr} \frac{\vec{r}}{r} = f'' \frac{\vec{r}}{r} \quad \vec{\nabla} \left( \frac{1}{r} \right) = -\frac{\vec{r}}{r^3} \quad \vec{\nabla} \cdot \vec{r} = 3$$

$$\Delta f = \frac{1}{r} f'' \frac{\vec{r}}{r} \cdot \vec{r} + f' \left( -\frac{\vec{r}}{r^3} \right) \cdot \vec{r} + 3 \frac{f'}{r} = f'' + \frac{2}{r} f'$$

$$3. g = f' \implies g' = f'' \implies \Delta f = g' + \frac{2}{r} g$$

$$\Delta f = 0 \iff g' + \frac{2}{r} g = 0$$

$$\implies \frac{dg}{dr} = -2 \frac{g}{r} \iff \frac{dg}{g} = -2 \frac{dr}{r}$$

$$\int \frac{dg}{g} = -2 \int \frac{dr}{r} \implies g = \frac{C}{r^2}$$

$$f(r) = C \int \frac{dr}{r^2} = -\frac{C}{r} + C_1$$

### Exercice 15

On considère les équations de Maxwell du champ électromagnétique

$$\begin{aligned} \vec{\nabla} \cdot \vec{H} &= 0 & \vec{\nabla} \times \vec{H} &= \epsilon \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{E} &= 0 & \vec{\nabla} \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \end{aligned}$$

$$1. \text{ Calculer } \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$$

$$2. \text{ Montrer que } \vec{E} \text{ vérifie l'équation d'onde } \Delta \vec{E} - \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \text{ avec } v = \frac{1}{\sqrt{\mu\epsilon}}.$$

$$3. \text{ Refaire le même calcul pour } \vec{H}.$$

$$4. \text{ Vérifier que } \vec{E} = \vec{E}_0 \exp(j(\omega t - \vec{k} \cdot \vec{r})) \text{ est une solution de l'équation d'onde, avec } \omega \text{ et } \vec{E}_0 \text{ constantes et } \vec{k} (k_x, k_y, k_z) \text{ tel que } k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\omega}{v}.$$



### Solution 15 :

$$1. \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$$

$$2. \text{ On a } \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \Delta \vec{E}$$

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \implies \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left( -\mu \frac{\partial \vec{H}}{\partial t} \right) \\ &= -\mu \vec{\nabla} \times \frac{\partial \vec{H}}{\partial t} = -\mu \frac{\partial \vec{\nabla} \times \vec{H}}{\partial t} = -\mu \frac{\partial}{\partial t} \left( \epsilon \frac{\partial \vec{E}}{\partial t} \right) = -\mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} \\ \vec{\nabla} \cdot \vec{E} &= 0 \implies \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) = \vec{0} \\ \implies \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= -\Delta \vec{E} = -\frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2}\end{aligned}$$

d'où :

$$\Delta \vec{E} - \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\begin{aligned}3. \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{H}) &= \vec{\nabla} (\vec{\nabla} \cdot \vec{H}) - \Delta \vec{H} = -\Delta \vec{H} \\ \vec{\nabla} \times (\vec{\nabla} \times \vec{H}) &= \vec{\nabla} \times \left( \epsilon \frac{\partial \vec{E}}{\partial t} \right) = \epsilon \frac{\partial \vec{\nabla} \times \vec{E}}{\partial t} \\ &= \epsilon \frac{\partial}{\partial t} \left( -\mu \frac{\partial \vec{H}}{\partial t} \right) = -\epsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2} = -\frac{1}{v^2} \frac{\partial^2 \vec{H}}{\partial t^2} = -\Delta \vec{H}\end{aligned}$$

$$\Delta \vec{H} - \frac{1}{v^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

$$4. \quad \vec{E} = \vec{E}_0 \exp(j(\omega t - \vec{k} \cdot \vec{r})) = \vec{E}_0 \exp(j(\omega t - xk_x - yk_y - zk_z))$$

$$\frac{\partial \vec{E}}{\partial x} = -jk_x \vec{E} \implies \frac{\partial^2 \vec{E}}{\partial x^2} = -k_x^2 \vec{E}$$

$$\text{de même : } \frac{\partial^2 \vec{E}}{\partial y^2} = -k_y^2 \vec{E} \text{ et } \frac{\partial^2 \vec{E}}{\partial z^2} = -k_z^2 \vec{E} \text{ et } \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$$

$$\Delta \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = -(k_x^2 + k_y^2 + k_z^2) \vec{E} = -k^2 \vec{E} = -\frac{\omega^2}{v^2} \vec{E} = \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

### Exercice 16

Les équations de Maxwell du champ électromagnétique dans le vide sont

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{E} &= 0 & \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}\end{aligned}$$

Soient  $\vec{A}$  un champ de vecteurs et  $\varphi$  un champ scalaire tels que :

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{et} \quad \vec{E} = -\vec{\nabla} \varphi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

1. Sachant que :  $\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0$ , démontrer que :  $\vec{A}$  et  $\varphi$  vérifient l'équation

$$\Delta \alpha - \frac{1}{c^2} \frac{\partial^2 \alpha}{\partial t^2} = 0$$

2. En un point  $M(x, y, z)$  on pose  $\vec{E} = E_x \vec{e}_x + E_y \vec{e}_y + E_z \vec{e}_z$  tel que :

$$\begin{aligned} E_x &= \frac{x}{x^2 + y^2} \cos(\omega t - kz) \\ E_y &= \frac{y}{x^2 + y^2} \cos(\omega t - kz) \\ E_z &= 0 \end{aligned}$$

où  $k$  et  $\omega$  sont deux constantes positives. et  $c = \frac{\omega}{k}$

Le champ magnétique en un point  $M(x, y, z)$  est défini par :

$$\vec{B} = \frac{k}{\omega} (E_y \vec{e}_x - E_x \vec{e}_y)$$

(a) Montrer que  $\vec{E}$  et  $\vec{B}$  vérifient les équations de Maxwell

(b) En utilisant la relation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \Delta \vec{E}$$

Montrer que  $\vec{E}$  vérifie l'équation de propagation d'onde :

$$\Delta \vec{E} + k^2 \vec{E} = 0$$



**Solution 16 :**

$$1. \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \text{ et } \vec{B} = \vec{\nabla} \times \vec{A} \text{ et } \vec{E} = -\vec{\nabla} \varphi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \implies$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A}$$

$$\frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{1}{c} \frac{\partial}{\partial t} \left( -\vec{\nabla} \varphi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right)$$

$\implies$

$$\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A} = \frac{1}{c} \frac{\partial}{\partial t} \left( -\vec{\nabla} \varphi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right)$$

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c} \frac{\partial \varphi}{\partial t} \text{ et } \frac{\partial}{\partial t} (\vec{\nabla} \varphi) = \vec{\nabla} \left( \frac{\partial \varphi}{\partial t} \right)$$

L'équation (2) devient

$$\vec{\nabla} \left( -\frac{1}{c} \frac{\partial \varphi}{\partial t} \right) - \Delta \vec{A} = -\frac{1}{c} \vec{\nabla} \left( \frac{\partial \varphi}{\partial t} \right) - \frac{1}{c^2} \frac{\partial \vec{A}}{\partial t} \text{ en simplifiant par } -\frac{1}{c} \vec{\nabla} \left( \frac{\partial \varphi}{\partial t} \right) \text{ on trouve}$$

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0 \quad (2)$$

$$\vec{\nabla} \cdot \vec{E} = 0 \implies \vec{\nabla} \cdot \left( -\vec{\nabla} \varphi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\implies -\Delta \varphi - \frac{1}{c} \vec{\nabla} \cdot \left( \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\Rightarrow \Delta\varphi + \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = 0 \Rightarrow \Delta\varphi + \frac{1}{c} \frac{\partial}{\partial t} \left( -\frac{1}{c} \frac{\partial\varphi}{\partial t} \right) = 0 \Rightarrow$$

$$\Delta\varphi - \frac{1}{c^2} \frac{\partial^2\varphi}{\partial t^2} = 0$$

$$2. \quad \vec{E} = E_x \vec{e}_x + E_y \vec{e}_y + E_z \vec{e}_z$$

$$(a) \quad \left. \begin{aligned} \frac{\partial E_x}{\partial x} &= \frac{\partial}{\partial x} \left( \frac{x}{x^2+y^2} \cos(\omega t - kz) \right) = \frac{-x^2+y^2}{(x^2+y^2)^2} \cos(\omega t - kz) \\ \frac{\partial E_y}{\partial y} &= \frac{\partial}{\partial y} \left( \frac{y}{x^2+y^2} \cos(\omega t - kz) \right) = \frac{x^2-y^2}{(x^2+y^2)^2} \cos(\omega t - kz) \\ \frac{\partial E_z}{\partial z} &= 0 \end{aligned} \right\}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

$$\left. \begin{aligned} \frac{\partial B_x}{\partial x} &= \frac{k}{\omega} \frac{\partial}{\partial x} \left( -\frac{y}{x^2+y^2} \cos(\omega t - kz) \right) = \frac{k}{\omega} \frac{2xy}{(x^2+y^2)^2} \cos(\omega t - kz) \\ \frac{\partial B_y}{\partial y} &= \frac{k}{\omega} \frac{\partial}{\partial y} \left( \frac{x}{x^2+y^2} \cos(\omega t - kz) \right) = -\frac{k}{\omega} \frac{2xy}{(x^2+y^2)^2} \cos(\omega t - kz) \\ \frac{\partial B_z}{\partial z} &= 0 \end{aligned} \right\}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{ky}{x^2+y^2} \sin(\omega t - kz) \vec{e}_x + \frac{kx}{x^2+y^2} \sin(\omega t - kz) \vec{e}_y$$

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} &= \frac{k}{\omega} \frac{\partial}{\partial t} ((-E_y \vec{e}_x + E_x \vec{e}_y)) = \frac{k}{\omega} \left( \left( -\frac{\partial E_y}{\partial t} \vec{e}_x + \frac{\partial E_x}{\partial t} \vec{e}_y \right) \right) \\ &= \frac{k}{\omega} \left( \frac{\omega y}{x^2+y^2} \sin(\omega t - kz) \vec{e}_x - \frac{\omega x}{x^2+y^2} \sin(\omega t - kz) \vec{e}_y \right) \\ &= -\vec{\nabla} \times \vec{E} \end{aligned}$$

$$\vec{\nabla} \times \vec{B} = -\frac{k^2}{\omega} \left( \frac{x}{x^2+y^2} \sin(\omega t - kz) \vec{e}_x + \frac{y}{x^2+y^2} \sin(\omega t - kz) \vec{e}_y \right) = \frac{k^2}{\omega^2} \frac{\partial \vec{E}}{\partial t}$$

$$(b) \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \Delta \vec{E}$$

$$\Leftrightarrow \vec{\nabla} \times \left( -\frac{\partial \vec{B}}{\partial t} \right) = \vec{\nabla} (0) - \Delta \vec{E}$$

$$\Leftrightarrow -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = -\Delta \vec{E}$$

$$\Leftrightarrow -\frac{\partial}{\partial t} \left( \frac{k^2}{\omega^2} \frac{\partial \vec{E}}{\partial t} \right) = -\Delta \vec{E}$$

$$\Leftrightarrow -\frac{k^2}{\omega^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\Delta \vec{E}$$

$$\Leftrightarrow -\frac{k^2}{\omega^2} (-\omega^2 \vec{E}) = -\Delta \vec{E}$$

Finalemment

$$\Delta \vec{E} + k^2 \vec{E} = 0$$

**Exercice 17**

En utilisant les dérivées en chaîne des fonctions composées, exprimer les opérateurs  $\vec{\nabla}$  et  $\Delta$  en coordonnées cylindriques.



**Solution 17** On a

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \iff \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan \frac{y}{x} \\ z = z \end{cases}$$

$$\text{Soit } g = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x}$$

$$r = \sqrt{x^2 + y^2} \implies \frac{\partial r}{\partial x} = \frac{1}{2} (2x) (x^2 + y^2)^{-1/2} = \frac{x}{r} = \cos \theta$$

$$\theta = \arctan \frac{y}{x} \implies \frac{\partial \theta}{\partial x} = \frac{-y/x^2}{1 + (y/x)^2} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2} = -\frac{\sin \theta}{r}$$

$$\frac{\partial z}{\partial x} = 0$$

$$g = \frac{\partial f}{\partial x} = \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta}$$

$$\text{Posons } h = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y}$$

$$r = \sqrt{x^2 + y^2} \implies \frac{\partial r}{\partial y} = \frac{1}{2} (2y) (x^2 + y^2)^{-1/2} = \frac{y}{r} = \sin \theta$$

$$\theta = \arctan \frac{y}{x} \implies \frac{\partial \theta}{\partial y} = \frac{1/x}{1 + (y/x)^2} = \frac{x}{x^2 + y^2} = \frac{\cos \theta}{r}$$

$$\frac{\partial z}{\partial y} = 0$$

$$h = \frac{\partial f}{\partial y} = \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta}$$

$$\begin{aligned} \vec{\nabla} f &= \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} = \frac{\partial f}{\partial r} \vec{e}_r + \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial f}{\partial z} \vec{k} \\ &= \left( \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta} \right) \vec{i} + \left( \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} \right) \vec{j} + \frac{\partial f}{\partial z} \vec{k} \\ &= \frac{\partial f}{\partial r} (\cos \theta \vec{i} + \sin \theta \vec{j}) + \frac{1}{r} \frac{\partial f}{\partial \theta} (-\sin \theta \vec{i} + \cos \theta \vec{j}) + \frac{\partial f}{\partial z} \vec{k} \end{aligned}$$

Soit en introduisant  $(\vec{e}_r, \vec{e}_\theta, \vec{k})$

$$\vec{\nabla} f = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial f}{\partial z} \vec{k}$$

**La divergence**

Soit  $\vec{H}$  un champ vectoriel défini, en coordonnées cylindriques, par

$$\vec{H} = R \vec{e}_r + T \vec{e}_\theta + Z \vec{k}$$

où  $R, T$  et  $Z$  sont des fonctions scalaires de  $(r, \theta, z)$ .

L'opérateur  $\vec{\nabla}$  s'écrit en coordonnées cylindriques :

$$\vec{\nabla} = \frac{\partial}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \vec{e}_\theta + \frac{\partial}{\partial z} \vec{k}$$

alors

$$\operatorname{div} \vec{H} = \vec{\nabla} \cdot \vec{H} = \left( \frac{\partial}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \vec{e}_\theta + \frac{\partial}{\partial z} \vec{k} \right) \cdot \vec{H}$$

Notons que  $\vec{e}_r$  et  $\vec{e}_\theta$  ne sont pas des vecteurs constants, ils sont des fonctions de  $\theta$  :

$$\begin{aligned} \vec{e}_r &= \cos \theta \vec{i} + \sin \theta \vec{j} \\ \vec{e}_\theta &= -\sin \theta \vec{i} + \cos \theta \vec{j} \end{aligned}$$

on a donc :

$$\frac{\partial \vec{e}_r}{\partial \theta} = \vec{e}_\theta \text{ et } \frac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{H} &= \frac{\partial \vec{H}}{\partial r} \cdot \vec{e}_r + \frac{1}{r} \frac{\partial \vec{H}}{\partial \theta} \cdot \vec{e}_\theta + \frac{\partial \vec{H}}{\partial z} \cdot \vec{k} \\ &= \frac{\partial R}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} (R \vec{e}_r + T \vec{e}_\theta) \cdot \vec{e}_\theta + \frac{\partial Z}{\partial z} \\ &= \frac{\partial R}{\partial r} + \frac{1}{r} \left( \frac{\partial R}{\partial \theta} \vec{e}_r + R \frac{\partial \vec{e}_r}{\partial \theta} + \frac{\partial T}{\partial \theta} \vec{e}_\theta + T \frac{\partial \vec{e}_\theta}{\partial \theta} \right) \cdot \vec{e}_\theta + \frac{\partial Z}{\partial z} \\ &= \frac{\partial R}{\partial r} + \frac{1}{r} \left( \frac{\partial R}{\partial \theta} \vec{e}_r + R \vec{e}_\theta + \frac{\partial T}{\partial \theta} \vec{e}_\theta - T \vec{e}_r \right) \cdot \vec{e}_\theta + \frac{\partial Z}{\partial z} \\ &= \frac{\partial R}{\partial r} + \frac{1}{r} \left( R + \frac{\partial T}{\partial \theta} \right) + \frac{\partial Z}{\partial z} \end{aligned}$$

Comme  $\frac{1}{r} \frac{\partial (rR)}{\partial r} = \frac{\partial R}{\partial r} + \frac{R}{r}$  on écrit :

$$\vec{\nabla} \cdot \vec{H} = \frac{1}{r} \frac{\partial (rR)}{\partial r} + \frac{1}{r} \frac{\partial T}{\partial \theta} + \frac{\partial Z}{\partial z} \quad (3)$$

### Le laplacien

Par définition :

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

On a :

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \cos \theta \frac{\partial g}{\partial r} - \frac{\sin \theta}{r} \frac{\partial g}{\partial \theta}$$

$$\frac{\partial g}{\partial r} = \frac{\partial}{\partial r} \left( \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta} \right) = \cos \theta \frac{\partial^2 f}{\partial r^2} - \frac{\sin \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\sin \theta}{r^2} \frac{\partial f}{\partial \theta}$$

$$\frac{\partial g}{\partial \theta} = \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta} \right) = -\sin \theta \frac{\partial f}{\partial r} + \cos \theta \frac{\partial^2 f}{\partial r \partial \theta} - \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} - \frac{\sin \theta}{r} \frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \cos \theta \left( \cos \theta \frac{\partial^2 f}{\partial r^2} - \frac{\sin \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\sin \theta}{r^2} \frac{\partial f}{\partial \theta} \right) - \frac{\sin \theta}{r} \left( -\sin \theta \frac{\partial f}{\partial r} + \cos \theta \frac{\partial^2 f}{\partial r \partial \theta} - \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} - \frac{\sin \theta}{r} \frac{\partial^2 f}{\partial \theta^2} \right)$$

$$\begin{aligned} &= \cos^2 \theta \frac{\partial^2 f}{\partial r^2} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial f}{\partial r} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} + \\ &\frac{\sin^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2} \end{aligned}$$

Soit :

$$\frac{\partial^2 f}{\partial x^2} = \cos^2 \theta \frac{\partial^2 f}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2} - 2 \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} + 2 \frac{\sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial f}{\partial r}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial y}{\partial y} = \frac{\partial h}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial h}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial h}{\partial z} \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 f}{\partial y^2} = \sin \theta \frac{\partial h}{\partial r} + \frac{\cos \theta}{r} \frac{\partial h}{\partial \theta}$$

$$\frac{\partial h}{\partial r} = \frac{\partial}{\partial r} \left( \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} \right) = \sin \theta \frac{\partial^2 f}{\partial r^2} + \frac{\cos \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} - \frac{\cos \theta}{r^2} \frac{\partial f}{\partial \theta}$$

$$\frac{\partial h}{\partial \theta} = \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} \right) = \cos \theta \frac{\partial f}{\partial r} + \sin \theta \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\cos \theta}{r} \frac{\partial^2 f}{\partial \theta^2} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta}$$

$$\frac{\partial^2 f}{\partial y^2} = \sin \theta \left( \sin \theta \frac{\partial^2 f}{\partial r^2} + \frac{\cos \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} - \frac{\cos \theta}{r^2} \frac{\partial f}{\partial \theta} \right) + \frac{\cos \theta}{r} \left( \cos \theta \frac{\partial f}{\partial r} + \sin \theta \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\cos \theta}{r} \frac{\partial^2 f}{\partial \theta^2} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta} \right)$$

$$= \sin^2 \theta \frac{\partial^2 f}{\partial r^2} + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial f}{\partial r} + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2} - \frac{\cos \theta \sin \theta}{r^2} \frac{\partial f}{\partial \theta}$$

Soit :

$$\frac{\partial^2 f}{\partial y^2} = \sin^2 \theta \frac{\partial^2 f}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2} + 2 \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} - 2 \frac{\sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial f}{\partial r}$$

Substitutions dans  $\Delta f$  :

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \cos^2 \theta \frac{\partial^2 f}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2} - 2 \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} + 2 \frac{\sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial f}{\partial r}$$

$$+ \sin^2 \theta \frac{\partial^2 f}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2} + 2 \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial \theta \partial r} - 2 \frac{\sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2}$$

Finalement :

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

L'expression de  $\Delta f$  en coordonnées cylindriques, peut être obtenue aussi de la définition :  $\Delta = \vec{\nabla} \cdot (\vec{\nabla} f)$

$$\Delta f = \left( \frac{\partial}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \vec{e}_\theta + \frac{\partial}{\partial z} \vec{k} \right) \cdot (\vec{\nabla} f) = \frac{\partial}{\partial r} (\vec{\nabla} f) \cdot \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} (\vec{\nabla} f) \cdot \vec{e}_\theta + \frac{\partial}{\partial z} (\vec{\nabla} f) \cdot \vec{k}$$

$$\frac{\partial}{\partial r} (\vec{\nabla} f) \cdot \vec{e}_r = \frac{\partial}{\partial r} \left( \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial f}{\partial z} \vec{k} \right) \cdot \vec{e}_r$$

$$= \left( \frac{\partial^2 f}{\partial r^2} \vec{e}_r + \frac{1}{r} \frac{\partial^2 f}{\partial r \partial \theta} \vec{e}_\theta - \frac{1}{r^2} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial^2 f}{\partial r \partial z} \vec{k} \right) \cdot \vec{e}_r = \frac{\partial^2 f}{\partial r^2}$$

$$\frac{1}{r} \frac{\partial}{\partial \theta} (\vec{\nabla} f) \cdot \vec{e}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial f}{\partial z} \vec{k} \right) \cdot \vec{e}_\theta$$

$$= \frac{1}{r} \left( \frac{\partial^2 f}{\partial \theta \partial r} \vec{e}_r + \frac{\partial f}{\partial r} \frac{\partial \vec{e}_r}{\partial \theta} + \frac{1}{r} \frac{\partial^2 f}{\partial \theta^2} \vec{e}_\theta + \frac{1}{r} \frac{\partial f}{\partial \theta} \frac{\partial \vec{e}_\theta}{\partial \theta} + \frac{\partial^2 f}{\partial \theta \partial z} \vec{k} \right) \cdot \vec{e}_\theta$$

$$= \frac{1}{r} \left( \frac{\partial^2 f}{\partial \theta \partial r} \vec{e}_r + \frac{\partial f}{\partial r} \vec{e}_\theta + \frac{1}{r} \frac{\partial^2 f}{\partial \theta^2} \vec{e}_\theta - \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_r + \frac{\partial^2 f}{\partial \theta \partial z} \vec{k} \right) \cdot \vec{e}_\theta = \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial}{\partial z} (\vec{\nabla} f) \cdot \vec{k} = \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial f}{\partial z} \vec{k} \right) \cdot \vec{k} = \frac{\partial^2 f}{\partial z^2}$$

d'où

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$